Incomplete Menu Preferences and Ambiguity

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Abstract: Problems of choice under uncertainty typically involve two stages, ex ante choice (i.e. the consumption commitment stage) and ex post choice (i.e. the consumption stage). The role of ex ante choice is to control ex post choice to the extent possible. Previous literature has studied this problem using the framework of acts, i.e. ex ante choices are complete schedules of ex post outcomes. This assumes that the decision-maker (DM) has more control over consumption outcomes than is typically feasible. By contrast, this paper analyzes choice problems in which the DM can only select second stage (ex post) outcomes up to noise. We model this by assuming the DM’s first stage choices are menus of outcomes as opposed to acts. Taking choices between menus as the observable, the paper presents three results. First, we provide a representation theorem which characterizes the DM’s choices between menus in terms of her beliefs about the second-stage selection process. Second, we identify the DM’s beliefs from first stage choices. Third, we provide comparative statics which link beliefs to bid-ask prices that would be associated to menus were they to be marketed.

Keywords: Menu Choice, Incomplete Preferences, Knightian Uncertainty.

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1 Introduction

This paper revisits a classical problem of choice under uncertainty. A decision-maker has to make a choice today which will determine a future consumption outcome. There is a temporal gap between the point of choice (call it time 0) and the subsequent point of consumption. Between these two points in time there is some payoff-relevant information, a “state of nature”, which is unresolved at the point of choice but which becomes known prior to the point of consumption. This gives rise to a time 0 betting market on how the state will resolve. Fixing the space of bets, the DM’s problem is to choose an optimal bet from those that are available to her.

In writing down a model of how this decision-maker makes choices, the analyst has to make some assumption about the available space of choices. The standard assumption that is made in the literature is the following: the DM’s choice set is the collection of all state-contingent payoff plans (hereafter, “acts”). In several classes of choice problems, this hypothesis assumes decision-makers have more control over outcomes than they actually possess in reality. This can, in turn, lead to distorted predictions about market behavior, particularly regarding the DM’s willingness to engage in speculative trade. The purpose of this paper is to conduct an analysis of this choice problem which relaxes this hypothesis.

We first present some examples in which the choice set is typically much smaller than the set of all acts, referring to these as “noisy choice” problems. A noisy choice problem has the following structure. First, we fix a set of consumption outcomes over which a DM has fixed, temporally invariant preferences. Next, instead of state-contingent schedules of outcomes (acts), we take the DM’s choice objects to be sets of outcomes (“menus”). Once a menu is chosen a specific outcome is randomly selected from this menu. When the source of this randomness has an unknown distribution,
i.e. it is subject to *uncertainty*, we interpret choice from the menu as noise since the DM is not fully in control of the consumption outcome. The best she can do is pick a menu, which then noisily maps to a selection from the menu.

This type of choice problem is frequently encountered in economics. Ahn (2008) gives an example of a patient (the DM) choosing among medical procedures, each of which has an uncertain outcome. This is a problem of noisy choice: identify procedures with subsets of (health) outcomes and note that the choice problem facing the DM can be recast as a selection from among these outcome sets. A second example is the problem of project investment, where an investor has to choose which among several projects to finance. As in the previous example projects are sets of outcomes, but the outcomes in this case are the possible ex post (cash) returns from the project. This too is a problem of noisy choice since the investor typically only gets to pick a set of return possibilities, as opposed to a specific return value.

For a third example, consider a problem of school choice. In this case, the DMs are the parents and “outcomes” are collections of characteristics which together describe an individual’s school experience, e.g. (good academic preparation, diverse peer group), and so on. In other words, a collection of characteristics jointly describes an outcome. Parents have fixed preferences over these outcomes, but they don’t get to directly choose these. All that is choosable is a set of outcomes, via school choice, and a particular outcome is then noisily selected from this set. The noisiness of educational outcomes is a substantive feature of the school choice problem. For example, empirical work which tests for the causal role of peer effects in educational achievement has relied on the presence of randomness in actual peer assignment mechanisms, e.g., roommate assignment in colleges (see Sacerdote (2001)).

Two features are common to each of these problems. The first is that choices are
menus and not acts. This is how choices are objectively presented to the decision-maker.\footnote{In a typical application, a noisy choice prospect would be modeled as a distribution over outcomes as opposed to a set of outcomes, i.e. where the set would be the support of the distributions. It is worth emphasizing that our model also interprets choices as distributions over outcomes. However, these distributions are also treated as subjective and the representation of a choice object as an implicit distribution over outcomes is derived from the primitive.} It may, nevertheless, be the case that a DM evaluates an outcome set by associating it to an act but, in such a case, the associated act is a subjective construct and not part of the primitive of the choice problem. That is to say, both the associated act and the process which connects a menu to an act must be derived from the observable. A second common feature is that, in each problem, choice is not precise. There is no uncertainty about consumption preference but there is a constraint on choices (i.e. the presence of noise) that prevents alignment of outcomes with preference. By contrast, if the choice space were the space of all acts, then all possible schedules of ex post outcomes would be feasible choices. This, in principle, allows the DM to more closely align outcomes to her fixed consumption preferences. However, in a noisy choice problem she can only select outcomes up to the uncertainty in nature’s choice. This implies her physical choice set is much smaller compared to a context in which she is choosing among acts.

The distinction is important. A sparser choice set can diminish willingness to participate in betting markets and explaining participation in betting markets is an important goal of the literature on choice under uncertainty. \cite{Bewley2002} (pg.80) notes that one of the most puzzling features about observed betting behavior is the extent to which it is largely absent outside of recreational venues, e.g. race-track betting. \cite{Bewley2002} also argues that models of Bayesian decision-making cannot explain the scarcity of betting. The argument he offers is that, other things equal, Bayesians will decline to bet against one another if and only if they have the same prior on how the state of nature will resolve. Since even modest levels of disagree-
ment would therefore predict more betting activity than is actually observed, an alternative model is needed to explain the absence of betting.

To this end, we develop a model with two features. First, it explains betting behavior that lies outside the Bayesian paradigm. Two bettors can have different beliefs about the state of nature yet decline to take bets against one another. The reason this happens is because, as in Bewley (1986), each DM has an associated set of beliefs and for two DMs to take bets against one another, participation must dominate abstention for every belief in their respective belief sets. This requirement, referred to as the “unanimity criterion”, was introduced in Bewley (1986) and provides a framework that has been adopted by many subsequent papers on this topic. All of these models are also based on the unanimity criterion, and thus can also explain non-Bayesian betting preferences. The second feature of our model is that we derive the unanimity criterion from a different observable than in the existing literature. In both the Bewley model and the subsequent work which builds on it, the observable is a preference relation over acts. In our paper, the observable is a preference relation over sets of outcomes (menus) as opposed to acts. This extends the unanimity criterion so that it applies to problems of noisy choice.

We present three main results. First, we formulate the unanimity criterion when the domain of choice is the set of menus and provide a set of axioms which characterizes this criterion. The main object to identify in our representation is the DM’s subjective belief set since this is what is responsible for the absence of trade. Our second result identifies subjective beliefs from the axioms. The third result concerns comparative statics. We consider a market where traders have subjective beliefs and all use the unanimity criterion to evaluate trades. Each menu available for trade is associated with a bid-ask spread of prices. Our third result shows that one DM exhibits, for each menu, a larger bid-ask spread than another if and only if she views
nature’s choice as more uncertain than the other, i.e. her subjective belief set contains the belief set of the DM who expresses tighter bid-ask spreads. Bid-ask prices are revealed through trading behavior, whereas subjective beliefs are elicited from preferences over menus. Hence, a corollary of the comparative static is that bid-ask prices are determined by and also determine a DM’s subjective beliefs. Put another way, prices revealed through trades allow us to fully recover the DM’s private information.

The remaining sections of the paper are organized as follows. In the next section we describe the model and the axioms. Section 3 presents the main results. These consist of (i) the representation theorem for the model, (ii) an identification result, and (iii) the comparative statics result described above. We then give a proof sketch of those steps in the representation that are novel to the menu choice domain. The remaining arguments are left to the appendix. Section 4 concludes and the appendix collects proofs.

1.1 Related Literature

This paper is most closely related to the literature on incomplete preferences. On account of our context being menu choice, it is also related to some of the literature on menu choice. We discuss each of these in turn, beginning with the literature on incomplete preferences. This is itself a vast literature, so we describe the initial papers along with more recent contributions which are most closely related to our paper: Aumann (1962), Bewley (1986), Dubra et al. (2004), Ghirardato et al. (2004), Nau (2006), Gilboa et al. (2010), Ghirardato and Siniscalchi (2012), Ok et al. (2012), and Galaabaatar and Karni (2013). Aumann (1962) is perhaps the first paper which gives a characterization of the unanimity criterion. His characterization applies the criterion to the choice domain of objective lotteries, i.e. the risk domain. Bewley (1986) generalizes this characterization to the domain of subjective lotteries.

We now describe some of the related work in (mostly) menu choice, i.e. Ghirardato (2001), Ahn (2008), and the results in Olszewski (2007), Chatterjee and Krishna (2008) (CK), Dekel and Lipman (2012) (DL), and Mihm and Ozbek (2018).\(^3\) Ghirardato (2001) and Ahn (2008) present different models to address the problem that the traditional primitive of acts imposes too much of a computational burden on the agent. Ghirardato (2001) takes as a primitive the set of coarse acts, i.e., functions mapping from states to sets of consequences (i.e., menus), and Ahn (2008) takes menus as a primitive and axiomatizes a model that bypasses the state-space technology.\(^4\) Of the latter group of papers, the first three axiomatize special cases of what is now referred to as the “random Strotz” model. When we add the completeness axiom to our list of postulates, our model collapses to this model. Olszewski

\(^3\)There is also work of Dekel et al. (2001) (DLR) on menu preferences. The DLR (“taste shock”) model is distinct from all of the models (including ours). However, one of the results in Dekel and Lipman (2012) shows an equivalence between a sub-class of taste shock models and a sub-class of models of the type we consider.

\(^4\)See section 2 where we discuss the connection between Ahn (2008) and our model in more detail.
(2007) axiomatizes the Hurwicz $\alpha$-maxmin criterion on the menu choice domain. CK axiomatize a generalization of Hurwicz maxmin where the “min” is replaced with a more general utility function. DL consider a model which they dub the “random Strotz” model. In their model choice from the menu is random, as in Olszewski (2007) and Chatterjee and Krishna (2008), but in contrast with Olszewski (2007) and Chatterjee and Krishna (2008) there is a continuum of states governed by a randomization process whose density function has a specific structure.\footnote{Dekel and Lipman (2012) define $U^{DL}(A) := \int_{v \in \mathcal{V}} \int_{\alpha} U_{v+\alpha \cdot u}(A) f(\alpha|v) d\alpha dv.$ (where $\mathcal{V}$ denotes a set of ex post states) and call this a \textit{continuous intensity random Strotz} representation. They show an observational equivalence between this model and the class of random Gul and Pesendorfer (2001) models. Under an additional continuity hypothesis on the function $U^{DL}(\cdot),$ they also provide an axiomatization of this model.}

\footnote{MO axiomatize $U(M) = \max_{\pi \in \Delta(S)} \{E_\pi \phi_M - c(\pi)\}.$ The term $\phi_M$ in the expectation in MO is the act induced by a menu and is related to the way in which our model converts menus to acts. For this reason, there is a connection between the complete preferences version of our model and the sub-class of the MO model corresponding to subjective expected utility (i.e. random Strotz). In this sub-case there is still a difference between the two models since our derived beliefs are necessarily probability measures (i.e. countably additive).}

5 Mihm and Ozbek (2018) (MO) axiomatize a generalization of the random Strotz model which is related to the variational preferences representation in Maccheroni et al. (2006) (MMR).\footnote{MO axiomatize $U(M) = \max_{\pi \in \Delta(S)} \{E_\pi \phi_M - c(\pi)\}.$ The term $\phi_M$ in the expectation in MO is the act induced by a menu and is related to the way in which our model converts menus to acts. For this reason, there is a connection between the complete preferences version of our model and the sub-class of the MO model corresponding to subjective expected utility (i.e. random Strotz). In this sub-case there is still a difference between the two models since our derived beliefs are necessarily probability measures (i.e. countably additive).}

2 Axioms and Model

Let $X := \{x_1, \ldots, x_k\}$ be a finite set and $\Delta(X) := \{(p_1, \ldots, p_k) : \sum_{i=1}^{k} p_i = 1, 0 \leq p_i \leq 1\}$ denote the set of lotteries on the set $X,$ viewed as a metric space with the topology induced by the usual Euclidean metric. Let $\mathcal{M} := \{M : M \subseteq \Delta(X), \ M \text{ closed}\}$ denote the space of \textit{menus} and let $d(\cdot, \cdot)$ be the metric induced by the Hausdorff topology (induced by the Euclidean metric). The behavioral primitive in this paper is a binary relation $\succeq$ on $\mathcal{M}.$
2.1 Model

The model is a version of Bewley’s Knightian uncertainty model which takes as an argument a menu, as opposed to an Anscombe-Aumann (AA) act. We dub it the subjective Bewley model. The model consists of:

1. A (subjective) state space $S$.

2. A set of (countably additive) probability measures $\Pi$ on $S$. We will suppress the domain of the measures in the description of the model, but for the purpose of comparative statics it will be important that both this domain (along with the measures themselves) are explicitly constructed from the axioms.

3. An affine utility function on consequences (lotteries) $u : \Delta(X) \to \mathbb{R}_+$. 

4. For each menu $M \in \mathcal{M}$ we associate a subjective act as follows:
   - For each state $s \in S$, let $u_s$ denote some vNM utility function on lotteries (not necessarily equal to $u$).
   - Let $M_s := \arg\max_{x \in M} u_s(x)$.
   - Let $\phi_M : S \to \mathbb{R}_+$ be defined by
     $$\phi_M(s) := \max_{x \in M_s} u(x).$$

We call $\phi_M(\cdot)$ a Strotzian value function. These functions form the space of subjective acts over which we define the Knightian uncertainty model. To this end, put

$$E_\pi \phi_M := \int_S \phi_M(s) \, d\pi_s$$

where $E_\pi \phi_M$ denotes the expectation of the Strotzian value function integrated with respect to the prior $\pi$ on $S$. Let $\Pi^7$ denote a set of priors on the space $S$ and define

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7We have neglected to mention the $\sigma$-algebra w.r.t. which the functions $\phi_M$ are measurable and how this is linked to the domain of the measures, e.g. it must be the case that the $\sigma$-algebra on which the functions are measurable is contained in the domain of the probability measures. We verify these details when we turn to the issue of identification.
the following Bewley-style unanimity criterion:

\[(*)\] \( M \succeq M' \iff E_\pi \phi_M \succeq E_\pi \phi_{M'} \ \forall \pi \in \Pi. \]

The subset of lotteries \( M_s \) should be thought of as the set of lotteries in \( M \) that are available in state \( s \). Hence, it is as if all other lotteries in \( M \) are infeasible in state \( s \) given nature’s selection. Once the state (i.e. nature in state \( s \)) “chooses” which lotteries are feasible in state \( s \) from the menu \( M \), the DM then maximizes his(her) state-independent preference over the set \( M_s \). Since the map \( M \mapsto \phi_M \) takes a menu to a function mapping from states to consequences we can think of \( \phi_M \) as a (subjective) Anscombe-Aumann act and the space of such \( \phi_M \) as a subspace of the set of all Anscombe-Aumann acts. It turns out that the set of Strotzian value functions is a nowhere dense subset of the set of all Anscombe-Aumann acts.\(^8\) Hence, while the DM converts a menu into an act he(she) has far fewer acts to choose from than in the setting where acts are observable.

This representation implies that the DM evaluates menus as distributions over ex post outcomes (within the menu). Put

\[ C_s(M) := \{ x \in M : x \in M_s, u(x) \geq u(y), \forall y \in M_s \} \]

equal to the set of state \( s \) choices (which is typically singleton when nature’s preference has no ties on \( M \)). Finally, note that using the model described above we obtain a reduced-form expression of menus as a distribution over outcomes supported on the menu, via:

\[ \pi^*(x) := \pi(\{ s \in S : x \in C_s(M) \}). \]

\(^8\)Regarding nowhere denseness, there is a minor issue regarding what topology to use on the space of Anscombe-Aumann acts. This is trivial with a finite state space, but we will find that – in general – our menu preferences will induce state space representations in which we genuinely require infinitely many states. The topology we use is the weak*-topology.
The measure $\pi$ then induces the probability distribution $\pi^*$ on $M$. To make sure this is well-defined when there are $u$-ties in $M$ we let $[x]$ denote the $u$-class of $x$ in $M$ and define the pull-back $\pi^*$ on classes $[x]$. This yields a well-defined probability distribution on the collection of $u$-classes $[x]$ in $M$. Let $E_{\pi^*}$ denote expectation w.r.t. the distribution $\pi^*$ and note that we have the equality,

$$E_{\pi^*} \phi_M = E_{\pi^*} u$$

where on the RHS the integration kernel is the utility function on lotteries. The Bewley model (with a single prior) shows one way in which menus are subjectively viewed as distributions.

We contrast this with the approach in Ahn (2008) which bypasses the state space technology and takes the distribution $\pi^*$ as a parameter in the model. That is, consider a pair $(u, \pi^*)$ where $\pi^*$ is a probability distribution on $\Delta(X)$ with $u$ a vNM utility on $\Delta(X)$, and let $\pi^*_M(\cdot)$ denote the conditional distribution on $M$ (where defined). Utility of the menu is then its $\pi^*_M$-expected value as above. An interesting question (which we do not answer here) is to find the conditions on the distribution $\pi^*$ that are necessary and sufficient for it to come from a state space model. The fact that we can express the state space model in the form $(u, \pi^*)$ suggests that there should be gap between the two approaches. The question, more concretely, would be to explicitly characterize this gap in terms of properties of the measure $\pi^*$.

2.2 Axioms

We first state some of the more standard axioms.

**Axiom 1**: (Pre-order) $\succeq \in \mathcal{M} \times \mathcal{M}$ is a transitive, binary relation with $\succeq|_{\Delta(X)}$ a complete and transitive relation.
Axiom 2a: (Independence) \( M \succeq M' \Rightarrow \alpha \cdot M + (1 - \alpha) \cdot M'' \succeq \alpha \cdot M' + (1 - \alpha) \cdot M'', \forall \alpha \in [0, 1] \).

Axiom 2b: (Indifference to Randomization) \( M \sim \text{co}(M) \).

This axiom is due to Dekel et al. (2001). Here \( \text{co}(M) \) denotes the convex hull of a menu \( M \). Both 2a and 2b are self-explanatory, so we omit motivation for these. They are also independent of each other.\(^9\) In the menu choice literature, independence together with the usual continuity axiom implies indifference to randomization (see Dekel et al. (2001)). However, the subjective Bewley model does not satisfy this continuity condition. We now introduce the continuity axiom that is implied by the model. First, introduce a bit of notation which will be used in other parts of the paper. Say that \( f \in M \) is non-redundant if \( f \notin \text{co}(M \setminus f) \). Say that a convex menu \( M \) is finitely convex if it can be recovered as the convex hull of finitely many of its non-redundant elements, i.e. \( M \) is a polytope.\(^{10}\) Recall that \( \text{relint}(M) \) denotes the relative interior of \( M \), where the ambient topological space is \( \Delta(X) \) and \( \text{ext}(M) \) denotes the set of extreme points of \( M \). The notation "\( M_n \to M \)" denotes convergence in the Hausdorff metric.

**Definition 1.** Fix a sequence of polytopes \( M_n \to M \). We say that convergence is regular if, for any \( f, g \ (f \neq g) \in \text{ext}(M) \) such that \( \text{co}(f, g) \cap \text{relint}(M) = \emptyset \) the following property holds: if \( f_n, g_n \in \text{ext}(M_n) \) with \( \text{co}(f_n, g_n) \cap \text{relint}(M_n) = \emptyset \) and

\(^9\)Models in Dekel et al. (2001) and Ergin and Sarver (2010) satisfy a stronger continuity axiom. In this case axiom 2b is a corollary of continuity, completeness, and independence. Dekel et al. (2007) dispense with stronger continuity but use completeness. Our menu preferences do not necessarily satisfy the stronger continuity condition or completeness.

\(^{10}\)In the context of our model, being finitely convex is the same being a polytope, but the notion of non-redundance is not, in general (i.e. without reference to state-space models), the same as being an extreme point. In deriving the representation, it will be useful to apply the more general notion (see lemmas 7,8).
In words, $M_n$ regularly converges to $M$ if (in addition to Hausdorff convergence) “boundaries of $M_n$ converge to boundaries of $M$” in the following sense: taking any two points on a boundary of $M$ the line spanned by these two points must equal the line spanned by any pair of convergents (i.e. pairs $p_n, q_n$ from $M_n$ converging to the two points in $M$) in $M_n$. In the appendix we show that this is equivalent to the statement that $\phi_{M_n}$ converges pointwise to $M$. A question that might come to mind having read this definition is: how do we know whether there are any regular sequences of menus? For details, we refer the reader to the section where we sketch the proof of the main representation. Briefly, the proof proceeds by building the representation on an increasing family of menus. For each family of menus the associated Strotzian value functions will turn out to be measurable w.r.t. a fixed partition of the subjective state space. Any convergent sequence of menus that are measurable with respect to the same partition will turn out to be regular. Moreover, in the proof, we will need to explicitly construct elements of the space of value functions measurable with respect to a fixed partition. Hence, there is a rich supply of regular sequences.

**Axiom 3**: (Continuity) If $M_n \succeq M'_n$ (resp. $M_n \preceq M'_n$) and $M_n \to M$, $M'_n \to M'$ are regular, then $M \succeq M'$ (resp. $M \preceq M'$).

A general continuity axiom would say that if “objects $A_n$ converge to object $B$” and “objects $A_n$ are all preferred to object $B$” (or vice-versa), then “the limit object maintains this preference”. Axiom 3 differs from this statement only in the requirement of convergence. In addition to the objects converging in the Hausdorff sense, they must be converging regularly. The axiom imposes a requirement only along such sequences, and is silent otherwise.
To state the next axiom we need to introduce some notation. First, let $\Theta(\succeq) := \{u \in \Delta(X) : u(\cdot) \text{ represents } (\succeq)|_{\Delta(X)}\}$. Note that these are lotteries whose associated linear functionals, i.e. $u(\cdot) : \Delta(X) \to \mathbb{R}_+$, represent the singleton ranking on menus. Let $\mathcal{E} := \{l \in \mathbb{R}^k : (i) \sum_i l_i = 0, (ii) u(l) = 0, \forall u \in \Theta(\succeq)\}$. Note that the set $\mathcal{E}$ is a set of translation vectors with the property that if we translate, say, a lottery $p$ by a vector $l \in \mathcal{E}$, then (assuming the translation remains a lottery) the translated lottery $p + l$ is in the same $(\succeq)|_{\Delta(X)}$-class as $p$.

**Axiom 4**: (Translation Invariance) Let $M, M' \in \mathcal{M}$ where $M' = M + l, l \in \mathcal{E}$. Then, $M \sim M'$.

For the final axiom, let $\inf(M), \sup(M)$ resp. denote the set of $(\succeq)|_{\Delta(X)}$-minimal (resp. maximal) lotteries in the menu $M$. While this particular axiom is from Stovall (2010), related axioms are used in Ahn (2008), Gul and Pesendorfer (2001), and Olszewski (2007).

**Axiom 5**: (Disjoint Set-Betweeness) If $\inf(M) \succeq \sup(M')$, then $M \succeq M \cup M' \succeq M'$.

Axiom 5 says that (i) we cannot make a menu better by adding lotteries that are worse (from the DM’s perspective) than anything already available and (ii) we make a menu worse by deleting its best lotteries. This set of axioms characterizes the model in this paper. A sketch of our construction is presented in the section following the statement of the main results.
3 Main Results

3.1 Representation Theorem

Definition 2. A binary relation on menus $\succeq \in \mathcal{M} \times \mathcal{M}$ admits a subjective Bewley representation if there is a quadruple $(u, \mathcal{S}, \mathcal{B}(\mathcal{S}), \Pi)$ consisting of (i) a vNM function on lotteries, (ii) a state space $\mathcal{S}$, (iii) a $\sigma$-field $\mathcal{B}(\mathcal{S})$ on $\mathcal{S}$, and (iv) a collection of probability measures $\Pi$ with domain $\mathcal{B}(\mathcal{S})$, that satisfies the following condition:

$$(*) \ M \succeq M' \Leftrightarrow E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi \in \Pi$$

where $\phi_M$ is the Strotzian value function (w.r.t. $\mathcal{S}$) associated to the menu $M$.

For brevity, call the quadruple $(u, \mathcal{S}, \mathcal{B}(\mathcal{S}), \Pi)$ a subjective Bewley model.

Theorem 1. A binary relation $\succeq \subseteq \mathcal{M} \times \mathcal{M}$ admits a subjective Bewley representation $(u, \mathcal{S}, \mathcal{B}(\mathcal{S}), \Pi)$ if and only if it satisfies Axioms 1-5 (i.e. 1,2a-b,3,4,5).

It deserves mention that the measures in the representation are probability measures, i.e. they are countably additive. This turns out to be important for comparative statics. In principle, the statement of our comparative static could be amended so that the measures in the representation are just finitely additive. However, we obtain countable additivity without imposing extra assumptions beyond our axioms. Countable additivity is also a common assumption in applied work. For example, in applications which analyze trading consequences of bid-ask spreads (see, e.g., Routledge and Zin (2009), Easley and O’Hara (2010), Ulrich (2013), Jeong et al. (2015), among many others), the asset return distributions which induce these spreads are assumed to be countably additive.

In a Savage framework, an additional hypothesis, e.g. the monotone continuity axiom (Villegas (1964)), is necessary to ensure that subjective beliefs are probability measures. We do not need such an axiom on our preferences. The reason is that,
in our model, there is a connection between states and ex post consequences, since “states” are just ex post choice functions.\textsuperscript{11} This allows us to discipline measures in our representation more than we could in a Savage framework without the monotone continuity axiom.

\textbf{Axiom 1'}: (Completeness) The binary relation $\succeq \subseteq \mathcal{M} \times \mathcal{M}$ is complete and transitive.

The following corollary generalizes three existing representations in the literature, (i) the $\alpha$-maxmin model in Olszewski (2007), (ii) the binary uncertainty (random) Strotz model in Chatterjee and Krishna (2008), and (iii) the “continuous intensity” random Strotz model in Dekel and Lipman (2012).

\textbf{Corollary 1}. A binary relation $\succeq \subseteq \mathcal{M} \times \mathcal{M}$ admits a subjective Bewley representation with $\Pi = \{\pi\}$ (i.e. there is a single prior) if and only if it satisfies Axioms 1', 2a-b, 3, 4, and 5.

\subsection*{3.2 Identification}

We now turn to uniqueness. There are three parameters of the representation derived from menu preferences, (i) the cardinal vNM utility on lotteries, $u$, (ii) the

\textsuperscript{11}A word about identification here. States are ex post functions in the particular model of preferences that we construct. However, this doesn’t by itself imply the same connection between states and state-dependent outcomes. For example, the definition of the subjective Bewley model doesn’t require that such a connection exists. Nevertheless, what our identification result shows is that, fixing any Bewley representation $(u, \mathcal{S}, \mathcal{B} \mathcal{S}, \Pi)$, we can (canonically) embed $\mathcal{S}$ in the state space that we construct in the proof of theorem 1 and also (canonically) extend the measures $\Pi$ to measures on this extended state space. This extension procedure is set-theoretic, and doesn’t use countable additivity. Hence, to check whether the original measures are countably additive it suffices to pass to the extended model, in which states are ex post choice functions, and check countable additivity for the extended model.
subjective state space and \( \sigma \)-field on this space, denoted \( (\mathcal{S}, \mathcal{B}(\mathcal{S})) \), and (iii) the measures \( \Pi \) on \( \mathcal{S} \) (with domain \( \mathcal{B}(\mathcal{S}) \)). Clearly \( u \) is identified up to affine equivalence since preferences over singleton menus satisfy the expected utility axioms. In our model, states are just index sets of ex post (expected utility) preferences. Hence, the individual state spaces, \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), are themselves subsets of the "universal" state space consisting of all vNM preferences, call this \( \mathcal{S}_{vNM} \).

Two facts follow from this. First, the set of Strotzian value functions measurable w.r.t. \( \mathcal{B}(\mathcal{S}) \) extend to functions measurable w.r.t. the Borel \( \sigma \)-algebra on \( \mathcal{S}_{vNM} \). Second, the measures \( \Pi \) push out to measures on \( \mathcal{S}_{vNM} \) (with domain the Borel \( \sigma \)-algebra on \( \mathcal{S}_{vNM} \)). Let \( \phi^* \) denote the push-out map. It follows that if we have two representations, say \((u_1, \mathcal{S}_1, \mathcal{B}(\mathcal{S}_1), \Pi_1), (u_2, \mathcal{S}_2, \mathcal{B}(\mathcal{S}_2), \Pi_2)\), then we can push out to get a pair of representations \((u_1, \mathcal{S}_{vNM}, \mathcal{B}(\mathcal{S}_{vNM}), \phi^*(\Pi_1)), (u_2, \mathcal{S}_{vNM}, \mathcal{B}(\mathcal{S}_{vNM}), \phi^*(\Pi_2))\). Uniqueness then reduces to showing that \( \phi^*(\Pi_1) = \phi^*(\Pi_2) \), which implies the equality, \( \Pi_1 = \Pi_2 \). The latter equality might seem a little strange since the domains \( \mathcal{B}(\mathcal{S}_1), \mathcal{B}(\mathcal{S}_2) \) are a priori not connected to one another. We view these measures as supported on the common state space \( \mathcal{S}_{vNM} \) and from this show that state spaces are unique up to sets of measure zero (for any measures in the set \( \Pi_1 = \Pi_2 \)). Uniqueness of sets of measures only holds (in our model) up to convexification. We suppress this in the statement of the result.

**Theorem 2.** Let \((u_1, \mathcal{S}_1, \mathcal{B}(\mathcal{S}_1), \Pi_1), (u_2, \mathcal{S}_2, \mathcal{B}(\mathcal{S}_2), \Pi_2)\) be two subjective Bewley representations of the same \( \succeq \subseteq \mathcal{M} \times \mathcal{M} \). Then, \( \Pi_1 = \Pi_2 \), \( \mathcal{S}_1 = \mathcal{S}_2 \) up to sets of \( \pi \)-measure zero, \( \forall \pi \in \Pi_1 = \Pi_2 \), and \( u_1 = au_2 + b \) (where \( a > 0 \)).

### 3.3 Comparative Statics

One of the reasons for the interest in the Bewley model is that it predicts a non-trivial bid-ask spread associated to Anscombe-Aumann acts. The mere presence of
this spread has some important implications for behavior in markets. As shown in Dow and da Costa Werlang (1992) and Rigotti and Shannon (2005), bid-ask spreads can account for the absence of trade and for indeterminacy of equilibria even when there is trade. Dow and da Costa Werlang (1992) analyze a single-agent portfolio choice problem and show that for asset prices within the bid-ask spread an agent will neither buy nor short the risky asset. Rigotti and Shannon (2005) analyze implications of uncertainty aversion in an Arrow-Debreu exchange economy. They characterize conditions on the sets of priors under which equilibria with trade exist and show that, unlike the standard model (with SEU preferences), these equilibria need not be determinate. They also point out that, while indeterminacy arises in other multiple prior models, e.g. the max-min expected utility (MEU) model in Gilboa-Schmeidler (Gilboa and Schmeidler 1989), the indeterminacies that arise in these cases are non-generic (in endowments), whereas they are generic when agents have Bewley preferences.

Menu preferences are revealed by observing choices between pairs of menus. In contrast, bid-ask spreads are revealed through trading behavior in markets. The purpose of this section is to connect these two pieces of choice data. In particular, we ask what bid-ask spreads reveal about a DM’s private information when the objects being priced are menus. Assume that all menu preferences admit a Bewley representation, denoted here in short via \((u, S, \Pi)\).

**Definition 3.** Fix a Bewley representation \((u, S, \Pi)\). For each menu \(M\), the bid-ask spread associated to \(M\) (w.r.t. the given representation) is defined as follows:

\[
\text{spread}_{(u, S, \Pi)}(M) := [\min_{\pi \in \Pi} E_{\pi} \phi_M, \max_{\pi \in \Pi} E_{\pi} \phi_M].
\]

In words, for each menu \(M\) we look at its Strotzian value function and compute its most pessimistic and optimistic values under the set of beliefs \(\Pi\). The quantity \(\text{spread}(M)\) denotes the interval of values between the most pessimistic and most
optimistic (expected) values of the menu. The lower end of this interval denotes the maximal price an agent would bid to acquire the (implicit) act represented by the menu \( M \) and the upper boundary denotes the minimal price the agent would ask to agree to a sale of \( M \). For this reason we refer to the interval of values as the “bid-ask” spread. The following result extends Proposition A.1 from Ghirardato et al. (2004) to the menu choice domain.

**Theorem 3.** Fix two models \((u_1, S_1, \Pi_1), (u_2, S_2, \Pi_2)\). Then,

\[
\text{spread}_{(u_1, S_1, \Pi_1)}(M) \succeq \text{spread}_{(u_2, S_2, \Pi_2)}(M), \quad \forall M
\]

if and only if \( u_1 = u_2, \Pi_1 \supseteq \Pi_2, \text{ and } S_1 = S_2 \) (up to \( \pi \)-null sets, for any \( \pi \in \Pi_1 \)).

As noted earlier, the bid-ask spread is revealed through trading behavior. The ask price is revealed by finding the minimal constant act (menu) the DM would accept in exchange for selling \( M \) and the bid price is revealed from the maximal constant act which the DM would trade in return for \( M \). A corollary of the theorem is that bid-ask prices determine the Bewley representation and, hence, both states and priors over states. Put another way, the bid-ask prices associated to each menu fully reveal a decision-maker’s private information.

On a formal level, theorem 3 is a menu-choice analogue of Proposition A.1 in Ghirardato et al. (2004). However, it deserves mention that the same proof does not apply. The reason is that with Anscombe-Aumann acts we can exploit, since Euclidean space is reflexive, a duality between probability measures and utility-valued acts. However, when the choice domain switches to menus this duality fails. In particular, there are Anscombe-Aumann acts which are not Strotzian value functions, so that there is no longer a natural correspondence between measures and choice objects when choice objects are menus. A formal consequence of the above theorem is that this duality is, in general, a stronger property than what is required or even implied by the comparative static. More concretely, the property that bid-ask spreads
uniquely determine the betting preference is much weaker than the requirement that there be a correspondence between bets and the priors which implicitly justify these bets.\footnote{The correspondence is via the duality map, where a bet $f_\pi$ corresponds to a prior $\pi$ if the expected value at $\pi$ of $f$ is higher (resp. lower) than its expected value under any other prior within the set of beliefs.}

### 4 Sketch of Proof of Theorem 1

We break the argument into the following steps: (i) Reduction, (ii) Transfer to ex post utility space, (iii) Representation on finitely measurable menus, (iv) Representation on all menus. The appendix has the details.

**Reduction and Transfer:**

We first define a state space of ex post preferences. Having done this, we then define a system of partitions of this state space that we call “Borel partitions”. Each menu maps to an act on this state space. As noted earlier, we call these menu-induced acts “Strotzian value functions”. The reduction step shows how we can group together menus whose associated Strotzian value functions are measurable with respect to the same Borel partition. This grouping has a natural order: as we refine the partition the set of menus (i.e. Strotzian value functions) measurable with respect to the refined partition enlarges. In this way we get an increasing filtration on the space of menus. The “limit” of this filtration will be the set of all menus. Transfer then shows that, when we fix the set of menus measurable w.r.t. any fixed partition in our pre-selected family of partitions, the preference over menus transfers to a preference over the utility acts which are induced by these menus.

Recall that $X$ is a finite set consisting of $k$ prizes, say $X = \{x_1, \ldots, x_k\}$, and

\[\text{...}\]
$\Delta(X)$ is the set of lotteries on $X$. The state space in this paper is (as a set) formally equal to the set of lotteries:

$$\mathcal{S} = \Delta(X) = \{(u_1, \ldots, u_k) \in \mathbb{R}^k : u_i \geq 0, \sum_i u_i = 1\}.$$\footnote{We take a different endogenous state space than typically used in the subjective state space literature, e.g. our state space is not the Dekel et al. (2001) state space. Fixing the set $\mathbb{R}^k$, where $k = |X|$, the Dekel et al. (2001) state space is defined as follows: $\mathcal{S}_{DLR} = \{s \in \mathbb{R}^k : \sum_i s_i = 0, \sum_i |s_i| = 1\}$. Under this parameterization of ex post preferences, states correspond one-to-one with vNM (ordinal) preferences over $\Delta(X)$. We are choosing a particular parametrization of a pre-image of the set $\Pi_{vNM}$.}

Note that, via the dot product, each lottery can be thought of an element of the dual space, say $p \mapsto p(\cdot)$. Each such dual element induces a vNM preference over $\Delta(X)$. Letting $\Pi_{vNM}$ denote the space of vNM preferences we obtain a surjective map $\mathcal{S} \mapsto \Pi_{vNM}$. That is, the map $p \mapsto p(\cdot)$ over-counts the set of vNM preferences.

Say that $s \triangleq s'$ if these states represent the same vNM preference on lotteries and let $[s] := \{s' \in \mathcal{S} : s' \triangleq s\}$. Let $\partial B$ denote the (topological) boundary of the set $B$.

**Lemma 1.** For each $s \in \mathcal{S}$, $\exists! s^* \in \partial \mathcal{S}$ such that $[s] = \{\alpha \cdot s^* + (1 - \alpha) \cdot \frac{1}{k} : \alpha \in (0, 1]\}$.

In other words, each vNM equivalence class of states is determined by a unique boundary state $s^*$ and is recovered by mixing (for an appropriate weight) $s^*$ with the degenerate vNM state $\frac{1}{k}$ (note that all lotteries are indifferent under $\frac{1}{k}$).

**Proof.** Taking any $s \in \mathcal{S}$, it is clear that there is some $s' \in \partial \mathcal{S}$ and $\alpha \in (0, 1]$ with $s = \alpha \cdot s' + (1 - \alpha) \cdot \frac{1}{k}$. Say that there is another boundary state $s'' \in \partial \mathcal{S}$ with $s'' \triangleq s$. By the Separating Hyperplane Theorem, there is an $\ell \in \mathbb{R}^{k}$ such that $\ell \cdot s'' \leq \ell \cdot \hat{s}, \forall s \in \{\alpha \cdot s' + (1 - \alpha) \cdot \frac{1}{k} : \alpha \in [0, 1]\}$ (with strictness for some $\hat{s}$). Since $s', s''$ are lotteries, we can – by adding $c\ell \ell$ to $\ell$ and dividing by an appropriate constant – assume wlog that $\ell$ is also a lottery. If $\ell \cdot s' > \frac{1}{k} \geq \ell \cdot s''$, then state $s'$ ranks lottery $\ell$ strictly higher than $\frac{1}{k}$ whereas state $s''$ ranks $\ell$ (weakly) lower. This contradicts the hypothesis that both $s'$ and $s''$ represent the same vNM preference on $\Delta(X)$. \hfill $\square$
Hence, consider $\ell.s', \ell.s'' \geq \frac{1}{k}$. Then, $\ell.(\alpha.s' + (1-\alpha)\frac{1}{k}) \geq \ell.s''$, $\forall \alpha \in (0, 1]$, implying that $\ell.s'' = 1/k$. This means the lotteries $\{\ell, \frac{1}{k}\}$ are indifferent under $s''$ but not under $s'$. For this case, take $\ell' := \frac{(1-\ell)}{k-1}$ and note that (i) $\ell'$ is a lottery whenever $\ell$ is and (ii) $\ell'.s \geq 1/k$ if and only if $\ell.s \leq 1/k$.

In our representation, derived beliefs will be defined on the synthetic state space defined above, but it is perhaps more intuitive to express these beliefs over the space of vNM preferences. Given beliefs on $\mathcal{S}$ we can equivalently view these as beliefs on the space of vNM preferences, $\Pi_{\text{vNM}}$, since for any event $E \in \mathcal{B}(\mathcal{S})$ we have $s \in E \iff [s] \in E$. This allows us to push-out measures on $\mathcal{S}$ to measures on $\Pi_{\text{vNM}}$. Concretely, let $\text{proj} : \mathcal{S} \Rightarrow \Pi_{\text{vNM}}$ denote the map, $\text{proj}(s) = [s]$. Given a $\sigma$-field $\mathcal{B}(\mathcal{S})$ on $\mathcal{S}$, we push-out to a $\sigma$-field on $\Pi_{\text{vNM}}$ via,

$$E \in \sigma(\Pi_{\text{vNM}}) \iff \text{proj}^{-1}(E) \in \mathcal{B}(\mathcal{S}).$$

The set of such $E$ is easily seen to be a $\sigma$-field (i.e. the push-out of the events on $\mathcal{S}$). Beliefs on $\sigma(\Pi_{\text{vNM}})$ are defined as follows. Given a probability measure $\pi$ with domain $\mathcal{B}(\mathcal{S})$ we put

$$\pi_{\text{vNM}}(E_{\Pi}) := \pi(\text{proj}^{-1}(E_{\Pi})).$$

This association clearly defines a probability measure on $\Pi_{\text{vNM}}$ and is one-to-one, i.e. no two $\pi$’s map to the same $\pi_{\text{vNM}}$. Hence, in terms of interpreting the model it makes no difference whether we use the parameterization $\mathcal{S}$ or $\Pi_{\text{vNM}}$. However, in terms of the analysis it will be more convenient to work with the space $\mathcal{S}$ directly in deriving beliefs. Figure 1 illustrates the state space in the case where $k = 3$, i.e. lotteries are supported on 3 prizes.

Having defined the state space $\mathcal{S}$ we map each menu $M$ to its Strotzian value function, $\phi_M$. When the menu $M$ is either finite or the convex hull of a finite set,
we can view $\phi_M$ as a vector in finite-dimensional Euclidean space. For such menus, when we group states in $\mathcal{S}$ according to the values assumed by the function, $\phi_M$, we obtain a particular kind of partition of the state space. We dub these partitions Borel partitions since they will end up generating the event space which forms the domain for the subjective priors. A (finite) Borel partition is a partition of the state space induced by finitely many hyperplanes. Formally, take a collection $\{ (p_i, c_i) \}_{i=1}^N$ where $p_i \in \mathbb{R}^k$ and group states in $\mathcal{S}$ according to which “side” of each hyperplane each state lies in, i.e. formally define an equivalence relation $\mathcal{R}$ on $\mathcal{S}$ via

$$s \mathcal{R} s' \iff [\forall i, p_i \cdot s \leq c_i \iff p_i \cdot s' \leq c_i]$$

The induced equivalence classes form what we call the Borel partition of $\mathcal{S}$.\textsuperscript{14}

Figure 2 describes a partition of this state space (in the case $k = 3$) defined by two hyperplanes, denoted by $\mathcal{L}_1, \mathcal{L}_2$. This partitions the state space $\mathcal{S}$ into four cells,\textsuperscript{14} States which lie on the boundary of some of the cells in this partition, i.e. which lie on one or more of the hyperplanes (call these “border” states), lie in separate cells than those which properly lie in each half-space. Partitions w.r.t which the functions $\phi_M$ are measurable will always have the property that border states are assigned the same $\phi_M$-value as certain interior states, so that our Borel partitions are actually coarsenings of the class of partitions described here. The specific manner in which the coarsening occurs turns out to be a subtle point which we relegate to the appendix.
Figure 2: An example of a Borel partition with \( k = 3 \).

resp. labelled \( E_1, \ldots, E_4 \). Now fix a given Borel partition of \( S \), call this \( \{E_i\} \). Let \( \Sigma_{E_i} \) denote the subset of all menus \( M \) whose associated Strotzian value function \( \phi_M \) is measurable w.r.t. the partition \( \{E_i\} \). The transfer step then identifies \( M \mapsto \phi_M \), so that we may think of \( \Sigma_{E_i} \) as a subset of ex post utility vectors in \( \mathbf{R}^{\{|E_i|\}}_+ \). Thinking of each vector \( \phi_M \) as an equivalence class of menus, we will need the set of utility vectors in \( \mathbf{R}^{\{|E_i|\}}_+ \) to inherit the axiomatic structure from \( \Sigma_{E_i} \). In the objective AA framework, this identification of acts with vectors of ex post (utility) payoffs is standard. However, in our case this is not a given. The reason is that we do not have an analogue of the monotonicity axiom on acts (which would explicitly involve states). The only axiom which explicitly relates ex post payoff consequences to the menu preference is Axiom 5. Hence, a non-trivial part of this step involves checking that the identification \( M \mapsto \phi_M \) respects the menu preference. This is the (only) part of the argument where Axiom 4 (Translation Invariance) is invoked.

**Representation on finitely measurable menus:**

\[\text{15There is a subtle point regarding how border states (i.e. states which lie on the hyperplanes themselves) are assigned to cells. It turns out that there is a canonical way to assign border states to cells, so that the partitions defined by hyperplanes are themselves canonical.}\]
This is the main step of the proof, in which we show that menus in $\Sigma_{E_i}$ admit a subjective Bewley representation. The main object to be elicited from the preference is the set of beliefs comprising the Bewley representation. To this end, there are two main steps in this argument. The first shows existence and uniqueness of a set of measures which gives a representation (this itself is broken up into two sub-steps, existence on each partition $\{E_i\}$ and "bootstrapping" across partitions to get a consistent and unique set of measures. The second shows positivity of these measures.

**Sub-step: Existence**

The initial steps here are completely standard. We let $C_{E_i}$ denote the pointed (convex) cone of functions $\phi_M$, i.e. $C_{E_i} := \{r \cdot (\phi_M - \phi_{M'}) : M \succeq M', r \in R_+\}$. Typically, one proceeds by showing that the cone is closed since we can then recover the cone as an intersection of its supporting hyperplanes. The normals to these hyperplanes are the candidate subjective beliefs that comprise the Bewley representation. When the choice domain is the full space of objective AA acts and the preference over these acts satisfies the monotonicity axiom, one then shows that these normals are positive multiples of probability measures. In our setting, the choice domain is the set of (menu-induced) Strotzian value functions and we do not have a monotonicity axiom on these value functions (since the axioms are inherited from the ones on the menu preference). For this part, the lack of monotonicity is less of a problem here than it might seem at the outset. The main obstacle is that the domain of menus and, hence, the domain of Strotzian value functions is *prima facie* much smaller than the set of all acts. To see why, note that if we are given any act $f(\cdot)$ and we modify to some $\hat{f}$ where $\hat{f}$ agrees with $f$ in all but one state, then $\hat{f}$ is obviously still an act and, hence, in the support of the preference relation.

When the support of the preference relation is the set of menus, this property
(which is vacuously true in the standard case) no longer holds. If we start with a Strotzian value function \( \phi_M \) and perturb its value in some states, then it is not necessarily the case that the perturbed function is also a Strotzian value function. To show this we would need to produce a perturbed menu whose associated Strotzian value function equals the perturbation of the original Strotzian value function. For a fixed Strotzian value function it is easy enough to come up with examples of perturbations of a given \( \phi_M \) such that the perturbed function is not a Strotzian value function.

One of the more involved steps in the proof is to show that there is some Strotzian value function such that all (sufficiently small) perturbations of this function are themselves Strotzian value functions. We show this by constructing a particular menu (called a “descriptive representation” of the partition \( \{E_i\} \)) whose perturbations provide an explicit basis of Strotzian value functions which span the full space \( R^{\left|\{E_i\}\right|} \). Since the set of Strotzian value functions (measurable w.r.t. \( \{E_i\} \)) is convex, this then implies that the set has non-empty interior and provides the candidate set of supporting measures which comprise the Bewley representation. This construction is carried out in lemmas 9-12, and proposition 2 in the appendix.

**Sub-step: Bootstrap**

In this sub-step we (i) construct consistent lifts of measures on a given partition to measures on the algebra of all (Borel) partitions, i.e. “bootstrap”, and (ii) show uniqueness of these lifts. The partitions \( \{E_i\} \) are partially ordered by the process of refinement, i.e. by adding more hyperplanes to the original partition. For each fixed partition \( \{E_i\} \) we have an associated subjective Bewley representation \( (\Sigma_{E_i}, \Pi_{E_i}) \), where \( \Pi_{E_i} \) is a set of measures on the (discrete) state space \( \{E_i\} \) (viewing each partition cell as a state). Let \( \Lambda \) denote an index set for the Borel partitions and abusing notation denote the partition \( \{E_i\} \) as \( E_\lambda \). We then have a collection of subjective Bewley representations \( (\Sigma_{E_\lambda}, \Pi_{E_\lambda})_{\lambda \in \Lambda} \). When \( E_\lambda \) refines \( E_\lambda' \) we have an embedding
\( \Sigma_{E_\lambda} \leftrightarrow \Sigma_{E_{\lambda'}} \), since functions \( \phi_M \) measurable w.r.t the coarsened partition are automatically measurable w.r.t refinements of this partition. Hence, since the measures \( \Pi_{E_{\lambda'}} \) restrict to measures on the coarsened partition \( E_\lambda \), uniqueness of the Bewley representation (which requires a separate proof for our model, see proposition 3) implies that the restricted set of measures agrees with \( \Pi_{E_\lambda} \). This implies that the collection \( \{ (\Sigma_{E_\lambda}, \Pi_{E_\lambda}) \} \) forms what is known as a compatible system. Defining \( \mathcal{A} \) to be the algebra (not \( \sigma \)-algebra) generated by all the finite partitions \( \{ E_\lambda \}_{\lambda \in \Lambda} \) we obtain a set of measures \( \Pi \) with domain \( \mathcal{A} \) whose restriction to any partition \( E_\lambda \) agrees with the set \( \Pi_{E_\lambda} \).

**Sub-step: Positivity**

What remains to be shown is that these measures are positive. This argument (as well as the existence argument) is referred to as “incentive compatibility” since it describes a method to control ex post (state-dependent) choices by ex ante menu design. The proofs are presented in steps 3iva-b of the appendix. Figure 3 below illustrates the main idea.

![Diagram illustrating incentive compatibility argument](image_url)

Figure 3: Incentive compatibility argument for steps 3iva-b, positivity of measures.
By reduction and transfer, states are grouped into (Borel) partitions so that positivity of measures amounts to showing positivity of these measures on cells of these partitions. Cells can be described (up to boundary restrictions) as cones cut out by the planes defining the partition. The figure above considers a special class of such partitions where cells are cones defined by a pair of planes. For such partitions states lying in the cone are organized along planes in which the “trade-off” rate of \( p - p' \) against \( r - r' \) is the same, where \( p - p', r - r' \) are normals to the planes bounding the cell (\( p', r' \) are companion lotteries to the lotteries \( p, r \), see, e.g., the proofs of lemmas 1 and 2 for a definition). States lying on the dotted line with lower slope have a larger MRS (i.e., a larger \( (1 - \gamma)/\gamma \) as defined in the picture). This means that, starting with a given lottery, shifts of this lottery towards the direction \( p - p' \) and away from \( r - r' \) are preferred less by such states than those that lie on a higher slope (smaller MRS). Put another way, state-wise utility differences between a lottery and its perturbation (where the perturbation is a convex combination of \( p - p', r - r' \)) are strictly monotone in the MRS. This fact, which can be thought of as a single-crossing condition, allows control over how states respond to perturbed choices when cells are formed by a pair of planes (as in the figure). More generally, the argument requires a set-wise, as opposed to pairwise, notion of the MRS (see step 3ivb).

**Representation on the full domain of menus:**

The final step of the proof is to show “by hand” that these measures are in fact countably additive and extends to all menus (i.e, that the representation extends to limits of \( \Sigma_{E_i} \)-measurable menus). Note that we don’t have a monotone continuity condition in our axioms. This is the usual vehicle through which SEU representations involving finitely additive beliefs are transferred to representations with countably additive beliefs. The reason we can do this is that we have a very explicit description of both the state space and the \( \sigma \)-field, i.e. it is just the \( \sigma \)-closure of the algebra \( \mathcal{A} \).
described in the previous paragraph. Hence, we check countable additivity by explicitly showing we can uniquely extend each of the measures to σ-limits of events in $\mathcal{A}$. This yields the representation on the domain of all finite consequence menus. To obtain the representation on the full domain of menus we use Axiom 3 (continuity) which allows us to characterize regularity in terms of pointwise convergence of the Strotzian value functions (see proposition 5). This allows approximation of infinite menus by finite menus and extends the representation to all menus, concluding the proof sketch.

**Additional comments:**

Dekel et al. (2001) (DLR) were among the first to use of menus of lotteries to construct state space models. Our (inductive) construction is different from the one in DLR and – as we argue below – necessarily so. DLR’s method first transfers preferences over menus to preferences over support functions, where the latter space inherits the linear structure of the menu preference. Support functions are dense in the space of all bounded and continuous functions, hence a cardinal, linear utility on menus – a priori defined only on the domain of support functions – can be uniquely extended to a linear function on the space of all (bounded, continuous) functions. Such functionals have an “integral characterization” as they are recovered by integrating the argument function against some measure $\mu$. This measure is the elicited subjective belief.

We also transfer preferences over menus to preferences over a space of functions, but in our case these are Strotzian value functions rather than support functions. After this point we make some changes – for two reasons. First, the DLR density argument does not apply to the cone generated by Strotzian value functions.\[16\]

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\[16\] The argument in Lemma 11, pg. 929, in Dekel et al. (2001) does not directly apply because it is not clear that set $\{r\phi_M : r \in \mathbb{R}_+\}$ of Strotzian value functions is closed under the supremum
ond, the continuity axiom implied by our model is not strong enough to ensure that extensions of the utility functional on the space Strotzian value functions possess an integral characterization.\footnote{These two facts together imply that the Dekel et al. (2001) procedure for extracting subjective beliefs will not work here without applying additional continuity conditions which are not implied by the model.\footnote{After Dekel et al. (2001), the first step in the procedure is extending, e.g. via a Hahn-Banach theorem, the utility functional on menus to a functional on the ambient function space. The second step is applying an integral characterization, e.g. a Riesz representation theorem, which characterizes the extended functional as an integral (expected utility).} It is for these reasons that we need to resort to our inductive construction to elicit beliefs.}

\section{Conclusion}

This paper studies a (static) model of choice under uncertainty in which the observable is a preference relation on menus. There are two-stages of choice but the DM is only involved in the choice between menus. Nature, and not the DM, makes the operation, i.e. $\phi_M \lor \phi_{M'}$ needs to be in this cone. For support functions, this follows by taking $M = M \cup M'$; however, unions of menus do not typically realize suprema of their respective Strotzian value functions, i.e. $\phi_M \neq \phi_M \lor \phi_{M'}$. Hence, to use the same approach one would need to produce for any pair $(M, M')$ a third menu $\hat{M}$ whose Strotzian value function is the pointwise supremum of $\phi_M$ and $\phi_{M'}$. Natural candidates, e.g. the union of $M, M'$, do not work.

\footnote{"Integral characterization" denotes the class of results, e.g. the Riesz representation theorem, which describe continuous linear functionals as integrals. The integral uniquely associates the functional with the measure against which we integrate the argument of the functional. There are (at least) two obstacles that must be overcome in finding an appropriate integral characterization. First, one must find an appropriate modification of a classical extension theorem, e.g. the Hahn-Banach theorem, that ensures the existence of a functional which both extends the linear functional on Strotzian value functions and which satisfies enough of a regularity condition for an integral characterization to apply. Second, and relatedly, one needs to find or prove an integral characterization result for the class of functionals which satisfy the regularity condition possessed by the extended functional.}
second stage choice from the menu. Moreover, nature’s choice is subject to uncer-
tainty so that ex ante it is as if an outcome is selected with noise. The DM evaluates
a menu ex ante as if it were an Anscombe-Aumann act, i.e. she constructs a state
space consisting of nature’s possible choice functions(preferences) and views a menu
as an act on this state space. The state space, the acts on this space, and the induced
preference relation on these acts are treated as unobservable and derived from ax-
ioms on menu choice. In this framework, we provide a representation theorem for an
analogue of Bewley’s unanimity criterion. In our model, uncertainty is represented
by the presence of multiple priors on the second stage choice process. Each prior
yields a different ex ante valuation of any given menu, so that multiple priors induce
a bid-ask spread for each menu. We provide comparative statics which show that
the collection of bid-ask spreads determines and is in turn determined by the DM’s
beliefs.
6 Appendix

Proof of Theorem 1

We proceed using the four step decomposition described in the main text. While subsequent steps build on earlier ones, the heart of the argument is step 3 where we construct the Bewley representation for menus which are measurable w.r.t a fixed Borel partition of the state space.

Step 1: Reduction

We first define three objects, (i) a subjective state space, (ii) a particular collection of partitions of this state space, and (iii) a collection of functions measurable w.r.t a partition of the state space (where the partition belongs to the collection of partitions we constructed).

Step 1i: Constructing a State Space

Put

\[ S := \{ (u_1, \ldots, u_k) \in \mathbb{R}^k_+ : \sum_{i=1}^k u_i = 1 \}. \]

Note that, on a formal level, the state space is just the simplex of lotteries on the finite set \( X \) of prizes, i.e. we may identify \( S \equiv \Delta(X) \). For each \( s \in S \) define a vNM utility on \( \Delta(X) \) by the formula \( U_s(p_1, \ldots, p_k) = \sum_{i=1}^k p_i u_i \). Clearly any given vNM preference is identified by an equivalence class of affine utilities. It is a simple exercise to check that for any affine utility \( U(\cdot) \), there are constants \( a, b \) with \( a > 0 \) such that the utility \( U' = aU + b \) has indices \( u_i \geq 0 \) and \( \sum_{i=1}^k u_i = 1 \). However, note that this process does not lead to a unique element of \( S \), so that \( S \) is a superset of the set of vNM preferences over \( \Delta(X) \). This turns out not to matter since (i) there is a well-defined projection map \( S \mapsto \Pi_{vNM} \) (where the latter is the set of ordinal vNM preferences over lotteries) and (ii) this map turns out to be measurable w.r.t.
the $\sigma$-field of events defined on $S$. Hence, when we construct the representation with
measures initially defined on $S$ we can (equivalently) express the representation in
terms of the pushed-out measure which lives on $\Pi_{vNM}$.

Step 1ii: Choosing partitions of the state space, $S$

As the space $S$ is a superset of the set, $\Pi_{vNM}$, of vNM preferences on $\Delta(X)$,
let $\Pi^*: S \rightarrow \Pi_{vNM}$ be the natural projection map defined by $\Pi^*(s) = \succeq_s$. That
is, the projection maps a state $s$ to the ordinal vNM preference generated by that
state. Put $[s] = (\Pi^*)^{-1}(s)$. We say that a partition of the state space $S$, denoted
generically with $\{E_i\}$, is coherent if for any $s \in S$, whenever $s \in E_i$, then $[s] \subseteq E_i$.
We now define a specific class of partitions of the state space.

Definition 4. A partition, $\{E_i\}$, of $S$ is called a Borel partition if it is generated
by a finite collection of hyperplanes $\{s: p_s \geq c_i\}$.

In the course of the proof we will define sets of measures for each partition of the
state space. As we refine the partitions, the sets of measures will also be refined.
The importance of the Borel property is that it allows us to show that the limiting
process, particularly the set of measures obtained in the limit, are independent of the
refinement process. On the other hand, we would also like to only consider coherent
Borel partitions since these are the only partitions that allow us to interpret the
results in terms of the set of ex-post vNM preferences.

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19 There is a subtle point involving what topology we put on the set $\Pi_{vNM}$. There are many
choices depending on how we choose to parameterize equivalence classes of cardinal representations
of a given $\succeq \in \Pi_{vNM}$. Given our $\sigma$-field on $S$ there turns out to be an obvious choice whose
associated open sets pull back to sets that are measurable w.r.t. to the $\sigma$-field we define on $S$.

20 The definition is incomplete since it does not clarify how to assign boundary states to partition
cells. This is step 2i below.
A Borel partition is specified by a collection of pairs \( \{(p_i, c_i)\} \) consisting of a separating vector together with a separating constant. One virtue of parameterizing the state space as we have is that we can equivalently describe the partition with a collection \( \{ (p^*_i, c^*_i) \} \), where \( p^*_i \in \Delta(X) \). That is, we can take the separating vectors to be lotteries. We refer to the collection \( \{ p^*_i, c^*_i \} \) as a *lottery representation* of the given Borel partition \( \{ E_i \} \). The following lemma gives a characterization of the set of Borel partitions in terms of their respective lottery representations.

**Lemma 2.** Let \( \{ E_i \} \) be a Borel partition of \( S \). The following are equivalent,

1. \( \{ E_i \} \) admits a lottery representation \( \{ p^*_i, c^*_i \} \), where \( c^*_i \equiv \frac{1}{k}, \forall i \).

2. \( \{ E_i \} \) is coherent.

**Proof of Lemma 2.** (2) \( \Rightarrow \) (1): Let \( \{ p_i, c_i \}_{i \in I} \) be a Borel partition that generates \( \{ E_i \} \) and note that \( \{ E_i \} \) is coherent if and only if the doubleton partition, \( S_1(i) := \{ s : p_1 \cdot s \geq c \}, S_2(i) = S \setminus S_1(i) \), is coherent for each \( i \in I \). Thus, we are reduced to the case of doubleton partitions, say \( S_1 = \{ s : p \cdot s \geq c \}, S_2 = \{ s : p \cdot s < c \} \). Note that we may normalize \( p \) such that \( p_i \geq 0 \) and \( \sum_i p_i = 1 \), so that lottery representations always exist. Via contraposition, if we have a lottery representation of the partition with \( c \neq 1/k \), say \( c > 1/k \). then consider

\[
s' := \alpha \cdot s + (1 - \alpha) \cdot \frac{1}{k}
\]

where \( s \in S_1 \) and \( \frac{1}{k} = (1/k, 1/k, ..., 1/k) \). Note that by coherence, \( s' \in S \). Choose \( \alpha \) sufficiently close to 0 so that \( p \cdot s' < 1/k + \epsilon < c \) - contradiction.

(1) \( \Rightarrow \) (2): As above we reduce to doubleton partitions \( S_1 := \{ s : p \cdot s \geq c \} \), \( S_2 = \{ s : p \cdot s < c \} \), where \( p \) is a lottery and \( c = 1/k \) by hypothesis. Define the following *companion* lottery, \( p' := \frac{(1-p)}{k-1} \) and consider the following coherent partition

\[p' \mapsto \frac{p + \varepsilon}{\sum_i (p_i + \varepsilon)} \]

in which the \( p^*_i \) are lotteries and we have for any \( s \in S \): \( p_i \cdot s \geq (\leq) c_i \Rightarrow p^*_i \cdot s \geq (\leq) c^*_i \).
of $S$,

$$E_1^* := \{ s : p \cdot s \geq p' \cdot s \}, \quad E_2^* := \{ s : p \cdot s < p' \cdot s \}$$

We claim that $S_1 = E_1^*$, $S_2 = E_2^*$. To see this note that

$$p \cdot s \geq p' \cdot s' = (\bar{1} - p) \cdot s = \frac{1}{k-1} - \frac{p \cdot s}{k-1}$$

so that $s \in E_1^*$ iff $p \cdot s \geq 1/k$ iff $s \in S_1$. $\square$

**Step 1iii: Measurable functions on $S$**

Fix a (coherent) Borel partition as $\{E_i\}$ and let $\Sigma_{E_i}$ denote the set of menus $M$ such that $\phi_M$ (the associated Strotzian value function is a $\{E_i\}$-measurable function (call a menu $\{E_i\}$-measurable when this happens). Let $\Lambda$ be an index of the set of all coherent Borel partitions (denote a generic partition as $\{E_\alpha\}$) and put

$$\mathcal{A} := \bigcup_{\alpha \in \Lambda} \{E_\alpha\}$$

Abusing notation, let $\Sigma_A$ be the set of all menus whose associated value functions, $\phi_M$, are $\mathcal{A}$-measurable (i.e. $\sigma(\phi_M^{-1}(B)) \subseteq \{E_i\}$ for some $\{E_i\} \in \mathcal{A}$). Thinking of $\phi_M$ as the ex post value function of the menu $M$, we now transfer the menu preference to preferences over these ex post value functions.

**Step 2: Transfer**

We transfer menu preferences to the space of induced ex post utility vectors, i.e. Strotzian value functions. The first sub-step of this argument clarifies the definition of the Borel partition and shows that the process of associating a menu $M$ to a Strotzian value function is well-defined (i.e. when the function is measurable w.r.t. a finite Borel partition). The second sub-step verifies that the affine relation on menus pushes out to an affine relation on $\{E_i\}$-measurable Strotzian value functions.
Step 2i: Mapping menus to Strotzian value functions

Let \( \{E_i\} \) denote a (finite) Borel partition of \( S \) and, abusing notation, let \( \{E_i\} \) also denote the \( \sigma \)-algebra on \( S \) generated by this partition. Let \( \Sigma_{E_i} \) denote the set of \( \{E_i\} \)-measurable menus, i.e. consider the function \( M : S \rightarrow \mathbb{R} \) defined by \( s \mapsto \max_{f \in M} u(f) \), where \( M_s := \arg\max_{f \in M} u_s(f) \). Consider the \( \sigma \)-algebra defined by \( \sigma(\phi^{-1}_M(B) : B \in \mathcal{B}) \), where \( \mathcal{B} \) is the Borel sigma-algebra on \( \mathbb{R} \). The menu \( M \) is said to be \( \{E_i\} \)-measurable if \( \sigma(\phi^{-1}_M(B) : B \in \mathcal{B}) \subseteq \{E_i\} \). Let \( k = |\{E_i\}| \) be the cardinality of the partition \( \{E_i\} \) and identify the set \( \Sigma_{E_i} \) with the equivalence classes of vectors in \( \mathbb{R}^k \) under the map into \( \mathbb{R}^k \) implicitly defined by \( M \mapsto \phi_M \). In defining this map, there is a subtle point concerning the borders of the partitions \( \{E_i\} \). That is, consider a pair of partition cells \((E_i, E_j)\) and take two sequences of states \( s^i_n \in \text{int}(E_i) \), \( s^j_n \in \text{int}(E_j) \) which converge to a common \( s^* \), which lies on the boundary of these two cells. Since the function \( \phi_M(\cdot) \) is, by hypothesis, \( \{E_i\} \)-measurable we must assign \( s^* \) to the cell \( E_i \) (resp. \( E_j \)) on which \( \phi_M \) obtains a higher value. But this then forces a consistency requirement on the way in which we assign border states to cells, viz. the assignment of the border state \( s^* \) to the higher value cell \( E_i \) must be the same for every \( \{E_i\} \)-measurable menu.

To see why this must be the case, think of the coordinates on the vectors in \( \Sigma_{E_i} \) as a space of symbols, \( \{1, 2, \ldots, n\} \) and, for brevity, denote the (finite-dimensional) utility vector \( \phi_M \) as \( u_M \). Our goal is to obtain a subjective Bewley representation of \( \Sigma_{E_i} \), which amounts to extracting a set of subjective beliefs on the synthetic state space \( \{1, 2, \ldots, n\} \). Our arguments in the forthcoming steps will imply the existence of a set of probability measures \( \Theta \subseteq \Delta(\{1, 2, \ldots, n\}) \) such that for any two vectors \( (u_M, u_M') \in \mathcal{U}_\phi \) (with \( u_M := (u_M(1), \ldots, u_M(n)) \) and \( u_M' := (u_M'(1), \ldots, u_M'(n)) \))
we have:

\[ u_M \geq u_{M'} \iff \sum_{i=1}^{n} u_M(i) \cdot \pi_i \geq \sum_{i=1}^{n} u_{M'}(i) \cdot \pi_i, \forall \pi \in \Theta. \]

However, the subjective Bewley representation requires that these measures are expressed over the state space \( S \). Hence, we pull back the measures in \( \Theta \) to the state space \( S \) via the map \( M \mapsto \phi_M \) to obtain (abusing notation, refer to a generic pullback as \( \pi \) as well)

\[ M \geq M' \iff \sum_{i=1}^{n} u_M(i) \cdot \pi(s : \phi_M(s) = u_M(i)) \geq \sum_{i=1}^{n} u_{M'}(i) \cdot \pi(s : \phi_{M'}(s) = u_{M'}(i)), \forall \pi \in \Theta. \]

The consistency condition we (implicitly) define above is the (necessary and sufficient) condition that allows this pull-back construction to be well-defined. First, note that in setting \( \pi(s : \phi_M(s) = u_M(i)) = \pi(i) \) we are implicitly assuming that the only states where the value of \( \phi_M \) equals \( u_M(i) \) are those states that comprise the partition cell \( E_i \). This means that if \( s \) is any state that lies at the border of two cells, say, \( (E_i, E_j) \), then if \( M \in \Sigma_{E_i} \) is such that \( u_M(i) = \phi_M(s) \geq u_M(j) \), then for any other menu \( M' \in \Sigma_{E_i} \) we must also have \( u_{M'}(i) = \phi_{M'}(s) \geq u_{M'}(j) \). Thus, consider the collection of interiors of the partition cells, \( \text{int}(E_i) \). The choice of a partition \( \{\hat{E}_i\} \) whose interiors correspond the collection \( \{\text{int}(E_i)\} \) implicitly determines an assignment rule on boundary states. The preceding sentence requires that we are able to find a partition with an assignment rule satisfying the consistency condition: (denote the border of a cell \( E_j \) as \( \partial E_j \))

**Consistency** If \( s \in E_i \) and \( s \in \partial E_i \cap \partial E_j \), then \( u_M(i) = \phi_M(s), \forall M \in \Sigma_{E_i} \).

We now verify this consistency condition by showing the existence of a canonical assignment rule that satisfies it.
Lemma 3. There is a canonical assignment rule $\kappa(\cdot)$ that satisfies consistency.

Proof of Lemma 3. Break the forthcoming argument into two pieces. First, reduce to the case where the collection of hyperplanes $\{L_1, \ldots, L_k\}$ which generate the partition all have the property that $s_u \notin L_i$, i.e. the state corresponding to the cardinal ranking on singletons, $u(\cdot)$, is in the interior of the partition. Second, verify that partitions formed by hyperplanes which contain $s_u$ are observably indistinguishable from those obtained by deleting these hyperplanes. For this reason, we refer to the steps (resp.) as (i) no unobservable hyperplanes and (ii) removal of unobservable hyperplanes.

Case (i): No unobservable hyperplanes.

Consider the following assignment rule. Abusing notation, let $\{\text{int}(E_i)\}_i$ denote the (disjoint) collection of sets formed by the intersections of the hyperplanes $L_1, \ldots, L_k$. We now specify a partition of $\mathcal{S}$ where the set of interiors of the partition cells is precisely the collection $\{\text{int}(E_i)\}$. We assign boundary states (i.e. states $s \in L_i$ for some $i$) as follows. For each state $s$ consider the set of states, $\{s'_\alpha : s'_\alpha := \alpha s + (1 - \alpha)s_u\}$. Recall that $s_u$ was the state whose vNM preference agrees with the normative ranking $u(\cdot)$. Notice that (since $s_u \notin L_i$) each $s'_\alpha$ is in $\text{int}(E_i)$ for some $i$. Assign the state $s$ to the unique $\text{int}(E_i)$ such that $s'_\alpha \in \text{int}(E_i), \forall \alpha \uparrow 1$. Note that if $s \in \text{int}(E_i)$, then $s$ gets assigned to $\text{int}(E_i)$ under this rule. This rule yields an assignment function,

$$\kappa : \mathcal{S} \to \{\text{int}(E_1), \text{int}(E_2), \ldots, \text{int}(E_n)\}$$

and put

$$E_i := \{s \in \mathcal{S} : \kappa(s) = \text{int}(E_i)\}$$

This defines a Borel partition. We now verify that it has the desired consistency property. To check this take a pair of adjacent cells $(E_i, E_j)$ (i.e. the cells share at least one hyperplane boundary) and let $M, M' \in \Sigma_{E_i}$. I claim that:

$$u_M(i) \geq u_M(j) \iff u_{M'}(i) \geq u_{M'}(j)$$
Find a state \( s \in \partial E_i \cap \partial E_j \) such that there is an \( \epsilon > 0 \) with \( B_\epsilon(s) \subseteq E_i \cup E_j \), i.e. the \( \epsilon \)-ball in \( S \) around state \( s \) lies entirely in the union of the two cells. Find a pair of sequences \( s^n_i, s^n_j \) such that \( s^n_i \in E_i, s^n_j \in E_j \) and \( s^n_i, s^n_j \to s \). Also find an associated pair of sequences of lotteries \( (f^n_i, f^n_j) \) such that \( f^n_i \in \arg \max_{f \in M} u_{s^n_i}(f), f^n_j \in \arg \max_{f \in M} u_{s^n_j}(f) \) and \( u(f^n_i) = u_M(i), u(f^n_j) = u_M(j) \). Put \( \Theta(i) := \{ s^n_i \}_n \cup \{ s \} \) and consider the correspondence \( \Phi(i) : \Theta(i) \supseteq \Delta(X) \) given by \( \Phi(i)(s) = \arg \max_{f \in M} u_s(f) \). Consider the graph of this correspondence, \( \text{Gr}(\Phi(i)) = \{(x, y) \in \Theta(i) \times \Delta(X) : y \in \Phi(i)(x) \} \) and, after passing to a convergent subsequence if necessary, we have \( (s, f^*) \in \text{Gr}(\Phi(i)) \) (where \( f^* := \lim f^n_i \)). Note that \( u(f^*) = u_M(i) \).

Now repeat the argument with the sequence \( s^n_j \) to obtain \( (s, f_*) \in \text{Gr}(\Phi(j)) \) where \( f_* := \lim f^n_j \) and \( u(f_*) = u_M(j) \). If \( \kappa(s) \in E_i \) (i.e. state \( s \) is assigned to cell \( E_i \)), then note that we must have \( u_M(i) \geq u_M(j) \). Find \( \alpha \in (0, 1) \) such that \( \alpha \cdot s_a + (1 - \alpha) \cdot s \in \text{int}(E_i) \). Towards contradiction, if \( u_M(i) < u_M(j) \) then take an alleged \( f_{s_a} \in \arg \max_{f \in M} u_{s_a}(\cdot) \) with \( u(f_{s_a}) = u_M(i) \). Note that:

\[
\begin{align*}
s_\alpha \cdot f_* &= \alpha s_a \cdot f_* + (1 - \alpha) s \cdot f_* \\
&> \alpha s_a \cdot f_{s_a} + (1 - \alpha) s \cdot f_* \\
&\geq \alpha s_a \cdot f_{s_a} + (1 - \alpha) s \cdot f_{s_a} \quad \text{(since \( f_* \in \arg \max u_s(\cdot) \)).}
\end{align*}
\]

This contradicts the assumption that \( f_{s_a} \in \arg \max u_{s_a}(\cdot) \). Thus, \( u_M(i) \geq u_M(j) \) whenever \( \kappa(s) \in E_i \). A symmetric argument using \( f^* \) shows that \( u_M(i) \leq u_M(j) \) whenever \( \kappa(s) \in E_j \). Since this conclusion holds for all menus in \( \Sigma_{E_i} \), the desired consistency property follows.

**Case (ii): Removal of unobservable hyperplanes**

From the preceding arguments we find that a (coherent) Borel partition can equivalently be described by a pair \( (\{ L_i \}, \gamma) \) consisting of a finite collection of hyperplanes, e.g. \( \{ s : p \cdot s = 1/k \} \), and an assignment rule \( \gamma(\cdot) \) that assigns border states to cells. Although \( \gamma \) need not be consistent in the sense defined above,
the state \( s \) must be assigned by \( \gamma(s) \) to a cell for which it is part of the border. That is, let \( \{ \text{int}(E_i), \ldots, \text{int}(E_i) \} \) be a list of cells for which \( s \in \text{int}(E_i) \). Then, we require that \( \gamma(s) \in E_i \) for one of these \( i \). Label the set of hyperplanes as \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_k, \mathcal{L}_{k+1}, \ldots, \mathcal{L}_n \} \), where \( s_u \notin \mathcal{L}_i, 1 \leq i \leq k \) and \( s_u \in \mathcal{L}_i, i \geq k + 1 \). Let \( \{ \mathcal{L}_i \}_{i=1}^k \) denote the Borel partition with assignment rule \( \kappa \) as defined above. Also note that the rule \( \gamma \) on the finer partition defined by the set \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_n \} \) induces an assignment rule on the (implicit) coarsening defined by \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_k \} \). Thus, starting with the consistent rule \( \kappa(\cdot) \) defined on the set of planes \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_k \} \), find an extension of this rule \( \gamma \) that gives an assignment rule on the full set of hyperplanes, \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_n \} \) (subject to the constraint specified above). Take \( \{ E_i \} \) to be the Borel partition generated by the pair \( (\{ \mathcal{L}_i \}_{i=1}^n, \gamma) \) and let \( \{ E_i' \} \) be the coarser partition generated by the pair \( (\{ \mathcal{L}_i \}_{i=1}^k, \kappa) \). Let \( M \in \Sigma_{E_i} \). We claim that, in fact, \( M \in \Sigma_{E_i'} \). To check this proceed by induction on the number of hyperplanes in the set \( \{ \mathcal{L}_{k+1}, \ldots, \mathcal{L}_n \} \), call this the set of “unobservable hyperplanes”.

**Base step:** \(|\{ \mathcal{L}_{k+1}, \ldots, \mathcal{L}_n \}| = 1 \).

Let \( \{ E_i \} \) denote the Borel partition formed by all \( k + 1 \) hyperplanes and let \( \{ E_i' \} \) denote the partition formed by \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_k \} \). Let \( \{ E_i', \ldots, E_i' \} \) be an enumeration of the partition cells that intersect the plane \( \mathcal{L}_{k+1} \). Also let \( E_{im}^1, E_{im}^2 \) denote the two cells into which the plane \( \mathcal{L}_{k+1} \) splits \( E_{im} \). Let \( s \in E_{im} \) and as above find a pair of sequences \( (s_n^1, s_n^2) \in E_{im}^1 \times E_{im}^2 \) such that \( s_n^1, s_n^2 \rightarrow s \). The same argument as above shows that \( \phi_M(s) \geq \phi_M(s_n^1) \) and \( \phi_M(s) \geq \phi_M(s_n^2) \). Say that \( \phi_M(s_n^1) \geq \phi_M(s_n^2) \). Note that for all \( \alpha \) close to 1 we have \( s_{\alpha} := \alpha s + (1 - \alpha)s_u \in \partial E_{im}^1 \cap \partial E_{im}^2 \). Moreover, note that we have \( u(f) \geq \phi_M(s), \forall f \in \arg \max u_{s_{\alpha}}(\cdot) \). Now apply the preceding closed graph argument again with \( s_n^2 \in E_{im}^2, s_n^2 \rightarrow s_{\alpha} \) to find an \( f^* \in \arg \max u_{s_{\alpha}}(\cdot) \) with \( u(f^*) = \phi_M(s_n^2) = \phi_M(s_n^1) \). Since \( u(f^*) \geq \phi_M(s) \) and \( \phi_M(s) \geq \phi_M(s_n^1) \) we obtain \( \phi_M(s_n^1) \geq \phi_M(s_n^2) \) implying that the value function \( \phi_M \) is constant on the union of the cells \( E_{im}^1 \cup E_{im}^2 \).
Inductive step:

For the inductive extension, take \( \{E'_i\} \) to be the partition generated by \( \mathcal{L}_1, \ldots, \mathcal{L}_k \) and \( m - 1 \) of the unobservable hyperplanes in the set \( \{\mathcal{L}_{k+1}, \ldots, \mathcal{L}_n\} \). Denote the partition generated by the original \( k \) hyperplanes and the \( m - 1 \) unobservable ones as \( \{E_i\} \). Add an additional unobservable hyperplane to the collection and let the corresponding partition be denoted as \( \{\hat{E}_i\} \). Mimic the base step argument to show that a menu \( M \in \Sigma_{E_i} \) descends to a measurable function on the coarser partition, so that \( M \in \Sigma_{E'_i} \). Now apply the induction hypothesis to claim that, in fact, \( M \in \Sigma_{E'_i} \).

This shows that there are no observable restrictions to omitting hyperplanes which contain the state \( s_u \) and concludes the proof (of the lemma) that there is a (canonical) consistent assignment rule on the boundary states; hence giving us a well-defined partition. \( \square \)

Step 2ii: Pushing the relation \( \succeq \subseteq \mathcal{M} \times \mathcal{M} \) out to \( \Sigma_{E_i} \times \Sigma_{E_i} \).

Having shown that there is a well-defined way to associate menu to a Strotzian value function, measurable w.r.t \( \{E_i\} \), we would now like to push out the binary relation on menus to a binary relation on the value functions (with the same axioms). Note that the map \( M \mapsto \phi_M \) is not necessarily one-to-one, viz. there may be menus \( M, M' \) with \( \phi_M = \phi_{M'} \). Hence, the only issue with transferring the binary relation out to the space of value functions is that it respects this equivalence. In other words, we need to show that if \( \phi_M = \phi_{M'} \) (and both \( M, M' \in \Sigma_{E_i} \)), then \( M \sim M' \). Once we show this, the axioms on menus transfer immediately to the space of value functions (note that the map \( M \mapsto \phi_M \) respects linearity, so that independence carries over without issue – the transfer of the other axioms being more transparent). This is the (only) step of the argument where we invoke Axiom 4 (viz. translation invariance).

**Proposition 1.** Fix any two menus \( M, M' \in \mathcal{M} \) with \( \phi_M, \phi_{M'} \in \Sigma_{E_i} \). Then, \( \phi_M = \)}
\(\phi_{M'}\) implies \(M \sim M'\).

**Proof of Proposition 1.** Break the argument into three (sub)steps (across lemmas 4-8). First, we prove the result for finite menus \(M, M'\) each of which is (i) \(\{E_i\}\)-measurable and (ii) \(\phi_M(s) \neq \phi_M(s'), \forall (s, s') \in E_i \times E_j\) (i.e. values assumed on all cells are distinct). Call such menus tightly \(\{E_i\}\)-measurable. Next, we extend the argument by relaxing this second condition (ii), and third, we extend to menus which are not necessarily finite.

**Lemma 4.** If \(M, M'\) are both finite, tightly \(\{E_i\}\)-measurable and \(\phi_M = \phi_{M'}\), then \(M \sim M'\).

**Proof of Lemma 4.** Let \(\{E_1, \ldots, E_k\}\) label cells in \(\{E_i\}\). Put \(M = \{f_1, \ldots, f_k\}\), and \(M' = \{f'_1, \ldots, f'_k\}\), labelled so that \(f_i\) (resp. \(f'_i\)) is chosen by states in cell \(E_i\). Consider a boundary plane of adjacent cells \((E_i, E_j)\), say with normal \(p_i - p'_i\). Consider the plane, \(\{s : (f_i - f_j) \cdot s = 0\}\) and note that it contains a boundary between \(E_i\) and \(E_j\). More precisely, note that given a cell \(E_i\) we can always find an adjacent cell \(E_j\) where the boundary formed by \((E_i, E_j)\) (denoted \(\partial E_i \cap \partial E_j\)) has the property that its linear span intersected with \(S\) equals the linear span of the plane on which the boundary lies intersected with \(S\). Put another way, the boundary \(\partial E_i \cap \partial E_j\) is a relatively open subset (in \(S\)) of the plane on which this boundary lies. Let \(p - p'\) denote a normal to the plane which equals the linear span of the boundary \(\partial E_i \cap \partial E_j\). Since \(\{s : (f_i - f_j) \cdot s = 0\}\) contains \(\partial E_i \cap \partial E_j\) it follows that \(f_i - f_j = \lambda \cdot (p - p')\), for some scalar \(\lambda\). Run through the same argument with \(M'\) to deduce that \(f'_i - f'_j = \lambda' \cdot (p - p')\). Note that \(u(f_i)(= u(f'_i)) \neq u(f_j)(= u(f'_j))\) by the fact that \(M, M'\) are tightly \(\{E_i\}\)-measurable. Hence \(u(p) \neq u(p')\). Since \(u(f_i) - u(f_j) = u(f'_i) - u(f'_j)\) we have \(\lambda = \lambda'\), so that

\[ f_i - f_j = f'_i - f'_j. \]

Equivalently,

\[ f_i - f'_i = f_j - f'_j. \]
Now find a connected sequence of cells $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$ where (i) $\{E_1, \ldots, E_k\} = \{E_{i_1}, \ldots, E_{i_k}\}$ and (ii) each pair $(E_{i_j}, E_{i_{j+1}})$ is separated by a boundary $\partial E_{i_j} \cap \partial E_{i_{j+1}}$ whose linear span is a hyperplane. It is straightforward to check, by induction on the number of hyperplanes comprising the partition $\{E_i\}$, that such a connected sequence always exists. The previous argument then yields the sequence of equalities:

$$f_{i_1} - f'_{i_1} = f_{i_2} - f'_{i_2} = \cdots = f_{i_k} - f'_{i_k}$$

Putting $\eta = f_{i_j} - f'_{i_j}$ (for any $j$) we find that $\sum \eta_i = 0$ and $u(\eta) = 0$ – and, hence, that $u(\eta) = 0, \forall u \in \Theta(\succeq)$ (see notation preceding statement of Axiom 4). Moreover, since $M' = M + \eta$, by Axiom 4 we have $M \sim M'$.

Lemma 5. Assume $M, M'$ are finite and $\{E_i\}$-measurable (but not necessarily tight). If $\phi_M = \phi_{M'}$, then $M \sim M'$.

Proof of Lemma 5. Even without tightness, we have the equality $f_i - f'_{i} = f_j - f'_{j}$ for adjacent cells $(E_i, E_j)$ on which the values of the common Strotzian value function, $\phi_M$, are distinct. Let $\eta(i)$ denote the translation on cell $E_i$. Tightness allows us to connect together all the translations on adjacent cells to assert that there is just a single translation. When we no longer have tightness, the preceding argument needs to be generalized. Find a maximal tight subset $M^* \subseteq M$. Given two adjacent cells $(E_i, E_j)$ in a partition $\{E_i\}$ we say that these cells are separated by a visible border if, letting $f_i, f_j$ denote the respective lotteries chosen in these cells, we have $u(f_i) \neq u(f_j)$. Introduce the following notation:

Definition 5. Fix a Borel partition $\{E_i\}$. Say that a sequence $(E_{i_1}, E_{i_2}, \ldots, E_{i_k})$ is a connected transversal of the partition $\{E_i\}$ if the following two properties hold:

1. Each cell $E_i$ in the partition $\{E_i\}$ appears on the list $\{E_{i_1}, E_{i_2}, \ldots, E_{i_k}\}$ at least once.

2. Consecutive cells on the list, viz. $(E_{i_j}, E_{i_{j+1}})$, are separated by a visible border.
Figure 4 gives a schematic representation of two Borel partitions (when $|X| = k = 3$). Colored cells denote the states in which (resp.) lottery $f_1, f_2,$ or $f_3$ are chosen. Note that the borders of the partition cells are “visible”, in the sense that there is a utility drop in the value of the function $\phi_M$ as we pass from one cell to the next.

Figure 4: A partition induced by $M = \{f_1, f_2, f_3\}$ where $u(f_1) \neq u(f_2) \neq u(f_3)$.

The next example (figure 5) shows how the partition can change when menus induce partitions without visible borders. The key step in the proof is to show that, while not all borders of a cell are visible, there will always be some border which is visible. This will allow us to connect cells across the partition, forming what we have dubbed a “connected transversal”. Since lotteries in any adjacent cells, say $(E_i, E_j)$, which share a visible border must exhibit the equality, $f_i - f'_i = f_j - f'_j$, a connected transversal allows us to string together these inequalities across every cell in the Borel partition. This then produces a common translate. The lemma below shows that a connected transversal exists, proving the indifference $M \sim M'$ in this case.

Consider $M = \{f_1, f_2, f_3\}$, where $u(f_1) = u(f_2) \neq u(f_3)$. Figure 5 describes the induced partition (for $k = 3$). Visible hyperplanes are solid. The hyperplane $L_1$ is
no longer visible, hence is suppressed.

Figure 5: A partition induced by $M = \{f_1, f_2, f_3\}$ where $u(f_1) = u(f_2) \neq u(f_3)$.

While the hyperplane $L_1$ is not observable, the choices made in the states carved out by the three planes $\{L_1, L_2, L_3\}$ are the same, viz. if $f_i$ is preferred to $f_j$ in state $s$ this preference is the same in the partitions induced by both menus. The only difference is that the cell of states in which $f_2$ is chosen no longer shares a visible border with the cell of states in which $f_1$ is chosen. Hence, while any path between cells in the first figure gives a connected transversal, in the above example there is only one way to get a connected transversal, viz. letting $E_i$ denote the cell of states in which $f_i$ is the choice from $M$ the unique connected transversal is $E_1 \rightarrow E_3 \rightarrow E_2$. This describes – in a very simple case – how we can construct a path through the cells of the partition, even when the menu $M$ is not tight. We therefore reduce to showing that $\Sigma_{E_i}$-measurable menus possess a connected transversal. 

**Lemma 6.** Assume $M$ (resp. $M'$) is finite and $\{E_i\}$-measurable. Then, $M$ (resp. $M'$) possesses a connected transversal.

**Proof of Lemma 6.** We check by adding lotteries one-by-one to $M^*$ that, along each step of the way, the resulting Borel partition admits a connected transversal. Prove this by induction. The previous step provides the base case, viz. if $M^* = M$. Now
Let $E_i^*$ denote a cell in $\{E_i^*\}$ in which $f_{ij}$ is chosen from $\{f_i\}$ and whose interior intersects the hyperplane, $\{s : (f_{ij} - f) \cdot s = 0\}$. Note that if $u(f_{ij}) = u(f)$, then $u(f_j) \neq u(f)$ as well. Let $E_{ij}^1, E_{ij}^2$ denote the two cells of the partition induced by $\{f_i\}$ in which (resp.) lottery $f_{ij}$ and $f$ are chosen. We claim that both $E_{ij}^1, E_{ij}^2$ must have a visible border with some adjacent cell in the partition induced by $\{f_i\}$. This is obvious for cell $E_{ij}^1$ (the one in which lottery $f_{ij}$ is chosen) since by the induction hypothesis it shares a border with some cell $E_j$ where a lottery $f_j$ is chosen with $u(f_j) \neq u(f_{ij})$. If the plane $\{s : (f_{ij} - f) \cdot s = 0\}$ intersects the plane $\{s : (f_{ij} - f_j) \cdot s = 0\}$, then it follows that the cell $E_{ij}^2$ has a visible border as well. Consider the case where $\{s : (f_{ij} - f) \cdot s = 0\}$ does not intersect $\{s : (f_{ij} - f_j) \cdot s = 0\}$ for all cells $E_{ij}^1$ which are cut by the plane $\{s : (f_{ij} - f) \cdot s = 0\}$. This implies the cell $E_{ij}^2$ has no visible border. We will now derive a contradiction from this. Let $\{g_1, \ldots, g_k\}$ denote the lotteries which are chosen in the cells which border $E_{ij}^2$ – where we assume $u(f) = u(g_i), \forall i$.

Pick any lottery $g \in \{f_i\}$ for which $u(f) \neq u(g)$ and define two convex sets: $K_1 := \{h : h = \alpha \cdot f + (1 - \alpha)g, \alpha \in (0, 1)\}$ and $K_2 := \text{con}(g_1, \ldots, g_k)$. Both sets are obviously convex and, by hypothesis, we have that $K_1 \cap K_2 = \emptyset$. By the separating hyperplane theorem, there is a non-zero $p \in \mathbb{R}^{|X|}$ such that

$$p \cdot k_1 \geq p \cdot k_2, \forall k_1 \in K_1, \forall k_2 \in K_2.$$  

By shifting $p$ to $p + c$ for $c > 0$ large and dividing by $\sum_i (p + c)_i$, we can (wlog) assume the separating vector is an element of $S$. This means that the vNM preference represented by $p$ ranks both $f$ and $g$ higher than lotteries in the border cells, viz. $\{g_1, \ldots, g_k\}$. If $f \succ_p g$, then find an (interior) state $s$ belonging to the cell in which $g$

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\[22\text{By maximality of } M^*, \text{ we know that } \{E_i^*\} \text{ must be a coarsening of } \{E_i^*\}.\]
is chosen (from $M^* \cup \{f\}$) and note that we have $g \succ_s f$. Find $\beta \in (0, 1)$ such that
$$\beta p(f - g) + (1 - \beta) s(f - g) = 0.$$ Consider the state $s' := \beta p + (1 - \beta) s$. Note that $s' g \geq s' g_i, \forall i$. Moreover, $s' g = s' f$, implying that there must be a visible border on cell $E^2_{ij}$ – contradiction. More precisely, what this argument shows is that there is some cell $E^*_{ij}$ (with an associated visible border cell $E^*_{ij}$) such that in the partition induced by $M^* \cup \{f\}$, both $E^1_{ij}$ and $E^2_{ij}$ share a common visible border with a cell, say, $E_j$ in which $f_j$ is selected. Here $E^2_{ij}$ denotes the cell in which $f$ is selected and $E^1_{ij}$ denotes the cell in which $f_j \in M^*$ is selected (when the option set is $M^* \cup \{f\}$). For reference in the forthcoming argument, label this pair of cells $(E^1_{ij}, E^2_{ij})$ and let (by abuse) $E^*_{ij}$ denote the cell in which lottery $f_j$ is chosen. Also let $E_{js}$ denote the cell which shares a visible border with each of $E^1_{ij}$ and $E^2_{ij}$.

We now extend the connected transversal $\{E^*_{i_1}, \ldots, E^*_{i_k}\}$ on the partition induced by $M^*$ to a connected transversal on $\{E^1_i\}$ (the partition induced by $M^* \cup \{f\}$) as follows. Abusing notation, let $E_i$ denote a generic cell in this partition. If $f_j$ (the choice in cell $E_j$) is not $f$ and $E_j$ is also a cell in the partition induced by $M^*$ (so that $E_j = E^*_j$) we leave the corresponding element of the sequence $\{E^*_{i_1}, \ldots, E^*_{i_k}\}$ unchanged. For cells $E_j$ where $f_j \neq f$ but where $E_j \neq E^*_j$ (so that the plane $\{s : (f_j - f) \cdot s = 0\}$ intersects this cell) we break into two cases. If $E_j \neq E^1_{ij}$, then we simply replace the element of the sequence $\{E^*_{i_1}, \ldots, E^*_{i_k}\}$ corresponding to states where $f_j$ is chosen with the corresponding cell $E_j$ (i.e. the smaller cell of states where $f_j$ is still chosen in the menu $M^* \cup \{f\}$). For the cell $E^1_{ij}$ replace the corresponding element in the sequence with $E^1_{ij}$. Moreover, insert $E_{js}$ as the next element in the sequence (note that it be making a second appearance here), followed by $E^1_{ij}$ and then continue with the original sequence, i.e. the next element is the associated consecutive element to $E^*_{ij}$ in the original sequence $\{E^*_{i_1}, \ldots, E^*_{i_k}\}$. We do this for every occurrence of the cell $E^1_{ij}$ in the original sequence $\{E^*_{i_1}, \ldots, E^*_{i_k}\}$. Note that this gives a connected transversal of the partition $\{E^1_i\}$. Now induct upwards to
using \( M^* \cup \{f\} \cup \{g\} \) as the base menu on which a connected transversal exists. Iteratively proceeding we obtain a connected transversal on \( M \).

One final comment on this case. The argument given above shows that for each pair \( \{f, f'\} \) of lotteries that are chosen by some state \( s \in \mathcal{S} \), we have \( f - f' = \eta \) for some common translate \( \eta \) with \( u(\eta) = 0 \). There may of course be elements \( f \) in the menu which are not chosen by any state, in which case we still need to relate these unchosen lotteries across \( M \) and \( M' \) to the translate \( \eta \). This is the purpose of the next step.

**Lemma 7.** Assume \( M, M' \) are (arbitrary) \( \{E_i\} \)-measurable. If \( \phi_M = \phi_{M'} \), then \( M \sim M' \).

**Proof of Lemma 7.** Call a lottery \( f \in M \) non-redundant if \( f \notin \text{con}(M \setminus f) \). When \( M \) is a finite menu, this is the same as saying \( f \notin \overline{\text{con}(M \setminus f)} \). Since both \( \{f\} \) and \( \overline{\text{con}(M \setminus f)} \) are closed convex (and disjoint) sets, there is some \( p \in \mathbb{R}^k \) with (\(*)\) \( p \cdot f > p \cdot f', \forall f' \in \overline{\text{con}(M \setminus f)} \). Adding a vector \( \vec{c} \) for some \( c > 0 \) large enough and dividing the inequality by \( \sum_i (p + \vec{c})_i \) we can take the separating vector to be a state \( s_f \in \mathcal{S} \). Hence, non-redundant lotteries are exactly those for which they are the unique element in the arg max for some state \( s \in \mathcal{S} \). Now (for finite \( M \)) redundant lotteries are exactly those that are convex combinations of non-redundant lotteries. Hence, if we show that non-redundant lotteries in \( M \) are translates of non-redundant lotteries in \( M' \) by some common \( \eta \), then any convex combination of non-redundant lotteries in \( M \) is also a translate, via the same \( \eta \), of a convex combination (with the same weights) of the corresponding translated non-redundant elements in \( M' \).

By the preceding paragraph (and indifference to randomization, viz. Axiom 2b), if there are only finitely many non-redundant elements, we then reduce the argument to the case of finite, \( \{E_i\} \)-measurable menus.

**Lemma 8.** Any \( \Sigma_{E_i} \)-measurable menu \( M \) has only finitely many non-redundant elements.
Proof of Lemma 8. Towards contradiction, say that $M$ possesses infinitely many non-redundant lotteries, $\{f_i\}$, yet is measurable w.r.t. a finite Borel partition $\{E_i\}$. Fix $f := f_1$. We claim that, via non-redundance, some infinite sub-collection, $\Lambda$, of the normal vectors $f - f_i$ must be non-parallel, viz.

$$f - f_i \parallel f - f_j, \forall i, j \in \Lambda.$$ 

Say that we have

$$f - f_i = \lambda \cdot (f - f_j).$$

If $\lambda > 0$ and $\lambda \leq 1$, then we have:

$$(1 - \lambda) \cdot f + \lambda \cdot f_j = f_i$$

which contradicts non-redundance of $f_i$. If $\lambda > 1$, then divide by $\lambda$ and apply the same argument verbatim to get $f_j$ as a convex combination of $f$ and $f_i$. Hence, we reduce to $\lambda < 0$. The preceding argument shows that, whenever the vectors $f - f_i, f - f_j$ are parallel, they must be negative scalar multiples of each other.

Since the collection $\{f - f_i\}$ is assumed to be infinite, find a triple $(i, j, k)$ with $f - f_i = \lambda_1 \cdot (f - f_j), f - f_j = \lambda_2 \cdot (f - f_k)$, with $\lambda_1, \lambda_2 < 0$. This implies

$$f - f_i = \lambda_1 \lambda_2 \cdot (f - f_k),$$

where the scalar multiple is positive. Apply the previous argument to contradict non-redundance. Hence, the planes normal to $\{f - f_i\}$ are all distinct. Note that if we take $\{f_i\}$ to be a list of the (by necessity, non-redundant) lotteries that are chosen by some state, then the Borel partition $\{E_i\}$ is recovered by the (by necessity, coherent) partition cut out by the planes with normals $\{f_i - f_j\}$. This means that if the partition $\{E_i\}$ is finite, then the set of equivalence classes of normal vectors, $f_i - f_j$, when grouped via scalar multiple form a finite set. But this means that infinitely many of the normals $\{f - f_i\}_{i \in \Lambda}$ must be equivalent mod scalar multiple, which cannot happen if each of the $f_i$ are distinct and non-redundant. \qed
Step 3: Representation on $\Sigma_{E_i}$

This step itself comprises of four sub-steps in total (Steps 3i-iv below). The first step is to show that the set, $\mathcal{U}_\phi$, of $\{E_i\}$-measurable Strotzian value functions has full dimension in the ambient vector space $\mathbb{R}^{\binom{|E_i|}{2}}$. This will then imply that the (pointed) cone $\{\phi_M - \phi_{M'} : M \succeq M', M, M' \in \Sigma_{E_i}\}$ has non-empty interior (since $\mathcal{U}_\phi$ is convex and finite-dimensional), so that the cone is closed. The candidate measures for the representation on $\Sigma_{E_i}$ will be the normals to the hyperplanes which support this (closed) cone. The second main step is to show that these measures are positive (i.e. not signed), so that after suitable normalization the normals can all be taken to be probability measures. Both of these are standard steps in derivations of ambiguity averse utility representations. However, while the steps to be proved are standard, how one is to go about proving them requires some non-standard arguments. The key obstacle is that in an Anscombe-Aumann setting any function mapping from states to consequences is in the choice set. In our notation, any act $f$ is of the form $f = \phi_M$. This is not true in our choice domain.

When choices are menus and we translate menus to acts via the map $M \mapsto \phi_M$ (i.e. mapping a menu to its Strotzian value function), the resulting space is a strict subset of the set of all Anscombe-Aumann acts. In the standard setting, checking both non-triviality of the (algebraic) interior of the pointed cone and positivity of the normals to hyperplanes supporting this cone follow trivially from the fact that any function is in the domain of the preference relation, e.g. so we can perturb the unit normal function in any direction and maintain an element of the pointed cone (so that the cone has non-empty interior), hence also verify that all measures are positive. On account of the fact that these perturbations do not prima facie exist in the set $\mathcal{U}_\phi$ we cannot simultaneously prove these two steps, and require a separate argument for each one.\footnote{Accordingly, we need to replace the unit normal with some other “canonical” choice of functions} To this end, we now introduce a special class of menus
Step 3i: Constructing Descriptive Representations

For notational brevity I will identify menus $M$ in $\Sigma_{E_i}$ with elements of $\mathcal{U}_\phi$, hence ignoring the distinction between a menu and its equivalence class under the map $M \mapsto \phi_M$. Let $\{E_i\}$ be the given Borel partition and let $\{(p_i, 1/K)\}$ be the associated lottery representation. Consider the set of pairs $(p_i, p'_i)$ consisting of the lottery and its companion. Say that a pair of cells $(E_i, E_j)$ in the Borel partition $\{E_i\}$ are adjacent if there is some lottery $(p^*_i, 1/K)$ that defines the partition such that the set $E_i \cup E_j$ is a cell in the Borel partition obtained by omitting this lottery. That is, $E_i \cup E_j$ is a cell in the Borel partition generated by the set of hyperplanes given by $\{(p_i, 1/K)\}\backslash(p^*_i, 1/K)$. I say a menu $M \in \mathcal{U}_\phi$ is a descriptive representation of the given Borel partition if it has the following properties,

1. $M = \{f_1, \ldots, f_k\}$, where $f_i = \sum_{j=1}^{k} a_j p^i(j), p^i(j) \in \{p_j, p'_j\}$
2. $0 \leq a_j \leq 1$
3. $f_i = \arg \max_{f \in M} u_s(f), \ \forall s \in \text{int}(E_i)$
4. $|\{j : p^i(j) \neq p'^i(j)\}| = 1$ for all pairs $(f_i, f'_i)$ where $(E_i, E'_i)$ are adjacent.

Lemma 9. For every Borel partition there is a menu which is a descriptive representation of that partition.
Proof of Lemma 9. Check this by induction on the number of hyperplanes that generate the Borel partition, i.e. the size of the set \( \{(p_i, 1/K)\} \). Note that the result is obvious when this set is singleton. Inductively assume we can construct a descriptive representation \( M \) whenever the set has size \( k - 1 \) or less. Now let \( \{(p_i, 1/K)\}_{i=1}^{k} \) denote a lottery representation of a Borel partition generated by \( k \) hyperplanes.

Note that this is obtained as the smallest refinement of the partition generated by \( \{(p_i, 1/K)\}_{i=1}^{k-1} \) and the partition generated by \( \{(p_i, 1/K)\}_{i=1}^{k} \). Let \( M_{k-1} \) be a descriptive representation of the partition generated by \( \{(p_i, 1/K)\}_{i=1}^{k-1} \) and define a new menu \( M_k \) as follows. Let \( \left\{(a_1, \ldots, a_{k-1}); (f_1, \ldots, f_{k-1})\right\} \) denote \( M_{k-1} \) in shorthand.

Let \( E_k(1), E_k(2) \) denote the partition of the state space induced by the lottery representation \( \{(p_i, 1/K)\}_{i=1}^{k} \), i.e. \( E_k(1) := \{s \in S : u_s(p_k) \geq u_s(p'_k)\} \), \( E_k(2) := \{s \in S : u_s(p_k) < u_s(p'_k)\} \). Also let \( \{E'_i\} \) denote the Borel partition induced by the set \( \{(p_i, 1/K)\}_{i=1}^{k-1} \). Put these partition cells into groups as follows. Define

\[ \Sigma_i := \{E'_j : \text{int}(E'_j) \subseteq E_k(i)\} \]

Pick any constants \( b_1, b_2 \) such that \( b_i > 0 \) and \( b_1 + b_2 = 1 \) and for each cell \( E'_j \) such that \( E'_j \in \Sigma_1 \cup \Sigma_2 \) define an associated lottery \( \hat{f}_j \) as follows.

\[
\hat{f}_j = \begin{cases} 
  b_1 \cdot f_j + b_2 \cdot p_k, & \text{if } E_j \in \Sigma_1 \\
  b_1 \cdot f_j + b_2 \cdot p'_k, & \text{if } E_j \in \Sigma_2 
\end{cases}
\]

For cells \( E_j \notin \Sigma_1 \cup \Sigma_2 \) define \( \hat{f}_j \) as follows. For \( s \in E_j \) let \( p_k(s) := \arg \max_{f \in \{p_k, p'_k\}} u_s(f) \). If the arg max is not singleton, let \( p_k(s) \) denote the lottery with the higher \( u(\cdot) \)-value.

Note that for each \( E_j \notin \Sigma_1 \cup \Sigma_2 \) we break up the cell into two new cells, \( E^1_j, E^2_j \) where \( \text{int}(E^1_j) \subseteq E_k(1), \text{int}(E^2_j) \subseteq E_k(2) \). Accordingly define adjusted lotteries \( \hat{f}_j \) as follows,

\[
\hat{f}_j = \begin{cases} 
  b_1 \cdot f_j + b_2 \cdot p_k, & \text{if } i = 1 \\
  b_1 \cdot f_j + b_2 \cdot p'_k, & \text{if } i = 2 
\end{cases}
\]
Put
\[ M_k := \bigcup_{j \in \Sigma_1 \cup \Sigma_2} \hat{f}_j \cup \bigcup_{j \notin \Sigma_1 \cup \Sigma_2} \{ \hat{f}_1^j, \hat{f}_2^j \} \]

Notice that by the induction hypothesis (and by construction) the menu \( M_k \) is a descriptive representation of the Borel partition generated by the set \( \{(p_i, 1/K)\}_{i=1}^k \). Therefore, all Borel partitions afford descriptive representations.

We now turn to the two sub-steps described in the text, (i) existence and (ii) showing that the individual measures patch together to give measures across all partitions. Existence is obtained in two sub-steps, (a) the existence argument for conical partitions and (b) the extension of this argument to non-conical, i.e. arbitrary, partitions. We will refer to the technique in the existence proof as a “perturbation argument”.

**Step 3ii: Existence of measures.**

**Step 3iia: Existence argument for conical partitions.**

I will now take a specific descriptive representation and show that one can generate a vector basis within \( U_\phi \) by taking suitable perturbations of this representation. The first step is to consider a special class of (coherent) Borel partitions – called conical partitions.

**Definition 6.** A Borel partition \( \{E_i\} \), with the additional property that the intersection \( L_i \cap L_j \) is identical for all pairs of hyperplanes \( L_i, L_j \) that define the partition is called a conical partition.

For some of the forthcoming arguments we will want to show that intersections of various pairs of hyperplanes agree. To show this, we will consider the hyperplanes (and their respective intersections) as living in the ambient vector space \( \mathbb{R}^k \) instead of the state space \( \mathcal{S} \). The following lemma shows that this is without loss. The
statement of the lemma holds for arbitrary finite intersections, though we state the result only for pairs.

**Lemma 10.** For any 4-tuple \((i, j, k, l)\) we have: \(\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{L}_k \cap \mathcal{L}_l \iff \mathcal{L}_i \cap \mathcal{L}_j \cap \mathcal{S} = \mathcal{L}_k \cap \mathcal{L}_l \cap \mathcal{S}\).

**Proof of Lemma 10.** To see this check the \(\Leftrightarrow\) implication via contraposition. Say that there is some \(s \in \mathbb{R}^k\) with \(s \in \mathcal{L}_i \cap \mathcal{L}_j \setminus (\mathcal{L}_k \cap \mathcal{L}_l)\). Since \(\mathcal{L}_i := \{s : (p_i - p'_i) \cdot s = 0\}\) (where each \(p_i\) is a lottery and \(p'_i\) is its companion), put \(\vec{c} = [c, \ldots, c]\) and note that \((p_i - p'_i) \cdot \vec{c} = 0\). Thus, find \(c > 0\) large enough so that \((s + \vec{c})_i > 0\) for all \(i\) and note that \((p_i - p'_i) \cdot (s + \vec{c}) \geq 0\)

Put \(\hat{s} := (s + \vec{c})/ \sum_{i=1}^k (s + \vec{c})_i\) to obtain that \(\hat{s} \in \mathcal{L}_i \cap \mathcal{L}_j \setminus (\mathcal{L}_k \cap \mathcal{L}_l)\).

When \(\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{L}_k \cap \mathcal{L}_l\) for all \((i, j, k, l)\) we call this a conical partition for the following reason. The intersection property implies that for any menu \(M\) which gives a descriptive representation \((M = \{(a_1, \ldots, a_n); (p_{i_1}, \ldots, p_{i_n})\})\) of the partition, the states that comprise, say, cell \(E_1\) are described by

\[E_1 = \{s \in \mathcal{S} : (i) \ p_{i_1}(1)s \geq p_{i_1}(1)'s, \text{ and } (ii) \ p_{i_2}(1)s \geq p_{i_2}(1)'s\}\]

That is, \(E_1\) is exactly the positive cone (in \(\mathcal{S}\)) formed by the two planes \(p_{i_1}(1) - p_{i_1}(1)’\) and \(p_{i_2}(1) - p_{i_2}(1)’\).

Let \(E_1, E_2, \ldots, E_k\) be an enumeration of the cells of the partition, labeled so that the boundary of \((E_i, E_{i+1})\) is given by \(\mathcal{L}_{i+1} := \{s : (p_{i+1} - p'_{i+1}) \cdot s = 0\}\). Assume the partition is generated by \(\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k\}\) and let \(M = \{(a_1, a_2, \ldots, a_k); (f_1, \ldots, f_{2+2k})\}\) be a descriptive representation.\(^{24}\) We label so that \(f_i\) is the lottery chosen by states in cell \(E_i\). Now consider the following perturbed lotteries. Put

\[\hat{f}_1 := f_1 + \alpha \cdot (p_1 - p'_1), \quad \hat{f}_2 := f_2 + \beta \cdot (p_3 - p'_3)\]

\(^{24}\)The size of the menu, \(2 + 2k\), turns out to be a consequence of the partition being conical, but this is an unimportant fact for what follows.
and, in addition, choose $\alpha, \beta$ such that (i) $u(\hat{f}_1) = u(\hat{f}_2)$ and (ii) $\alpha, \beta$ are sufficiently small, where the definition of "sufficiently small" is to be defined below. Let $\hat{M} := \{\hat{f}_1, \hat{f}_2, f_3, \ldots, f_{2+2k}\}$. Consider the following hypothesis:

**(H):** There is a pair of states $(s_1, s_2) \in E_1 \times E_2$ with $\hat{f}_1 \succ s_1 \hat{f}_2, \hat{f}_2 \succ s_2 \hat{f}_1$.

Let us assume for the time being that we can choose perturbed lotteries $\hat{f}_1, \hat{f}_2$ such that hypothesis (H) holds. We now check that this implies that the menu $\hat{M}$ is $\Sigma_{E_i}$-measurable. We then use this to show that the span of the set $\mathcal{U}_\phi$ has full dimension. Consider the hyperplane $\mathcal{L} := \{s : \hat{f}_1 \cdot s = \hat{f}_2 \cdot s\}$.

**Lemma 11.** The partition obtained by adding the plane $\mathcal{L}$ is also conical.

*Proof of Lemma 11.* To see this, label the planes that define the borders of cells $E_1$ and $E_2$ as $p_n, p_1, p_2$. That is, $p_n - p'_n$ defines the border between cells $E_n$ and $E_1$, $p_1 - p'_1$ defines the border between cells $E_1$ and $E_2$, etc. Note that we are making strong use of the conical structure here since we can associate cones formed by border planes with partition cells in a one-to-one manner. Let $\mathcal{L}_i$ denote the plane corresponding to the border formed by the lottery $p_i$. We first claim that

$$\mathcal{L}_2 \cap \mathcal{L}_n \subseteq \mathcal{L} \cap \mathcal{L}_2$$

To see this note that $\mathcal{L}_2 \cap \mathcal{L}_n = \mathcal{L}_1 \cap \mathcal{L}_n = \mathcal{L}_1 \cap \mathcal{L}_2$. Therefore, if $s \in \mathcal{L}_2 \cap \mathcal{L}_n$, then $s \in \mathcal{L}$. The containment follows. Now notice that $\mathcal{L}_2 \cap \mathcal{L}_n$ and $\mathcal{L} \cap \mathcal{L}_2$ are both linear spaces of dimension one less than the dimension of the hyperplane. Since strict subspaces must have strictly smaller (linear) dimension, it follows that $\mathcal{L}_2 \cap \mathcal{L}_n = \mathcal{L} \cap \mathcal{L}_2$. Now observe that we have the following string of equalities,

$$\mathcal{L} \cap (\mathcal{L}_2 \cap \mathcal{L}_n) = \mathcal{L}_2 \cap \mathcal{L}_n$$

$$\mathcal{L} \cap (\mathcal{L}_i \cap \mathcal{L}_j) = \mathcal{L}_i \cap \mathcal{L}_j \text{ (since } \mathcal{L}_2 \cap \mathcal{L}_n = \mathcal{L}_i \cap \mathcal{L}_j)$$

The latter implies that $\mathcal{L}_i \cap \mathcal{L}_j \subseteq \mathcal{L} \cap \mathcal{L}_i$. Comparing linear dimensions yields the equality $\mathcal{L} \cap \mathcal{L}_i = \mathcal{L}_i \cap \mathcal{L}_j$. Therefore, the augmented partition is also conical.  \[ \square \]
Let \( \{E_1, \ldots, E_n\} \) denote the cells in the original partition and assume that \( E_i, E_j \) are such that \( \mathcal{L} \cap \text{int}(E_i) \neq \emptyset \) and \( \mathcal{L} \cap \text{int}(E_j) \neq \emptyset \). Let \( \{(p_1 - p'_1), \ldots, (p_n - p'_n)\} \) denote the set of normal vectors associated with the collection of hyperplanes, \( \{\mathcal{L}_1, \ldots, \mathcal{L}_n\} \), that form the partition. Let \( \{(a_1, \ldots, a_n); (f_1, \ldots, f_n)\} \) be a descriptive representation of the partition, where \( f_i = \sum_{j=1}^n a_j \cdot f_i(j) \) (and \( f_i(j) \in \{p_j, p'_j\} \)). Let \( \text{supp}(f_i) := \{f_i(1), \ldots, f_i(n)\} \).

**Lemma 12.** If \( \mathcal{L} \) intersects both \( \text{int}(E_i) \) and \( \text{int}(E_j) \), then \( \text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset \).

**Proof of Lemma 12.** Check this via contradiction. If there is some \( f_* \in \text{supp}(f_i) \cap \text{supp}(f_j) \), find \( g_* \notin \text{supp}(f_i) \cap \text{supp}(f_j) \) (which exists since \( E_i \neq E_j \)) and let \( g_* \in \text{supp}(f_i), g'_* \in \text{supp}(f_j) \). Take \( s_i \in \mathcal{L} \cap \text{int}(E_i) \) and \( s_j \in \mathcal{L} \cap \text{int}(E_j) \) and note that

\[
s_i \cdot (g_* - g'_*) > 0 \text{ and } s_j \cdot (g_* - g'_*) < 0
\]

Find \( \alpha \in (0, 1) \) so that \( \alpha [s_i \cdot (g_* - g'_*)] + (1 - \alpha) [s_j \cdot (g_* - g'_*)] = 0 \). Consider the state \( s_\alpha := \alpha s_i + (1 - \alpha) s_j \) and note that \( s_\alpha \cdot (g_* - g'_*) = 0 \). Thus, letting \( \mathcal{L}_\alpha \) denote the hyperplane in the collection \( \{\mathcal{L}_1, \ldots, \mathcal{L}_n\} \) with normal \( (g_* - g'_*) \), we have \( s_\alpha \in \mathcal{L}_\alpha \). Since \( s_\alpha \in \mathcal{L} \) we obtain, as the (augmented) partition is conical (lemma 11)

\[
s_\alpha \in \mathcal{L} \cap \mathcal{L}_\alpha = \mathcal{L}_i \cap \mathcal{L}_j, \forall \mathcal{L}_i, \mathcal{L}_j \in \{\mathcal{L}_1, \ldots, \mathcal{L}_n\}
\]

Thus, \( s_\alpha \cdot (p_i - p'_i) = 0, \forall i \). However, there is some \( f_* \in \text{supp}(f_i) \cap \text{supp}(f_j) \), which implies (find \( p_k \) such that \( f_* \in \{p_k, p'_k\} \)) that either

1. \( s_i \cdot (p_k - p'_k) > 0, s_j \cdot (p_k - p'_k) > 0 \) or
2. \( s_i \cdot (p_k - p'_k) < 0, s_j \cdot (p_k - p'_k) < 0 \).

In either case, we obtain \( s_\alpha \cdot (p_k - p'_k) \neq 0 \) – contradiction. \( \square \)

It follows that if \( (E_i, E_j) \) are such that \( \mathcal{L} \cap \text{int}(E_i) \neq \emptyset \) and \( \mathcal{L} \cap \text{int}(E_j) \neq \emptyset \), then \( \text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset \). Now consider a state \( s \in \mathcal{S} \) where, say, \( \hat{f}_1 \in \arg \max_{f \in M} u_s(f) \) (strictly). Put \( E_1 \cup E_2 = \{s : (p_1 - p'_1) \cdot s \geq 0, (p_3 - p'_3) \cdot s \geq 0\} \) and recall \( \mathcal{L} := \{s \in \mathcal{S} : (\hat{f}_1 - \hat{f}_2) \cdot s \geq 0\} \).
Proposition 2 (Incentive Compatibility).

1. Put \( C(L_1, L) := \{ s \in S : (p_1 - p'_1)s \geq 0, (\hat{f}_1 - \hat{f}_2)s \geq 0 \} \). Then, \( C(L_1, L) \subseteq E_1 \cup E_2 \).

2. Put \( L' := \{ s : (\hat{f}_2 - \hat{f}_1)s \geq 0 \} \) and define \( C(L_3, L') := \{ s : (\hat{f}_2 - \hat{f}_1)s \geq 0, (p_3 - p'_3)s \geq 0 \} \). Then, \( C(L_3, L') \subseteq E_1 \cup E_2 \).

The label “incentive compatibility” comes from interpreting states as preference types and considering, for any state \( s \), whether the choice from the menu \( \hat{M} \) is different from the choice from \( M \). Observe that if there is a change in choice going from \( M \) to \( \hat{M} \) then the states at which this change occurs must be in one of the cones, \( C(L_1, L), C(L_3, L') \). To see this, if the switch is to \( \hat{f}_1 \), then \( \hat{f}_1 \) is weakly preferred to \( \hat{f}_2 \), i.e. \( s \cdot (\hat{f}_1 - \hat{f}_2) \geq 0 \). Moreover, if \( f_j \) was the initial choice at state \( s \) (from the menu \( M \)) and \( \hat{f}_1 \) is now in the arg max, then it must be the case that \( s \cdot (p_1 - p'_1) \geq 0 \), else \( f_1 \) is itself preferred (strictly) to \( \hat{f}_1 \). Hence, all such states are in the cone \( C(L, L_1) \). Similar reasoning shows that any state for which \( \hat{f}_2 \) is in the arg max (from the menu \( \hat{M} \)) is in the cone \( C(L_3, L') \). Hence, if we show that these two cones are in the union of the cells \( E_1 \cup E_2 \), then nature’s choice at all states outside these two cells is unchanged as we perturb menu \( M \) to \( \hat{M} \).

Proof of Proposition 2. (1). It suffices to check that \( \{ s : s \cdot (p_1 - p'_1) \geq 0, s \cdot (\hat{f}_1 - \hat{f}_2) > 0 \} \subseteq (E_1 \cup E_2) \). Towards contradiction, if this containment did not hold, then there would be an element of \( s \in C(L_1, L) \) with \( s \notin E_1 \cup E_2 \). Hence, \( s \cdot (p_1 - p'_1) \geq 0, s \cdot (\hat{f}_1 - \hat{f}_2) > 0 \) and \( s \cdot (p_3 - p'_3) < 0 \). By hypothesis (H) and the fact that the augmented partition is conical (by lemma 11), we obtain that the plane \( L \) intersects

\[ \text{Note that any } s \text{ in the cone is a limit of such points. Moreover, since the original partition is conical we have that } E_1 \cup E_2 \text{ is itself a (closed) cone. Hence, the limit points are in the union as well. This presumes that the union of the cells } E_1 \cup E_2 \text{ contains the boundary of the cones which border these cells, under the canonical assignment map } \kappa(\cdot). \] For the set of cases in which we apply this proposition this will always be true. We will be concerned with boundary states at a later point in the argument, but we state the modifications as the need arises.
the interior of the union of cells $E_1$ and $E_2$. Hence, find another state $\hat{s}(\not\in E_1 \cup E_2)$ such that:

i. $\hat{s} \cdot (p_1 - p'_1) > 0$,

ii. $\hat{s} \cdot (\hat{f}_1 - \hat{f}_2) < 0$, and

iii. $\hat{s} \cdot (p_3 - p'_3) < 0$.

Find $\alpha > 0$ such that, putting $s^* := \alpha \cdot s + (1 - \alpha) \cdot \hat{s}$, we have $s^* \cdot (\hat{f}_1 - \hat{f}_2) = 0$, so that $s^* \in \mathcal{L}$. Note that $s^* \not\in E_1 \cup E_2$ (since $s^* \cdot (p_3 - p'_3) < 0$). Let $E_i(i \neq 1, 2)$ denote the cell to which $s^*$ belongs. By the fact that the original partition is conical, let $p(i_1) - p(i_1)'$, $p(i_2) - p(i_2)'$ be such that $E_i = \{s : s \cdot (p(i_1) - p(i_1)') \geq 0, s \cdot (p(i_2) - p(i_2)') \geq 0\}$. Note that the inequalities defining $\hat{s}$ are all strict. Hence, there is an open ball of states around $\hat{s}$ for which the same inequalities hold. Moreover, for each state, say $s'$, in the ball there is some $\alpha_{s'}$ such that $(\alpha_{s'} \cdot s + (1 - \alpha_{s'}) \cdot \hat{s}) \cdot (\hat{f}_1 - \hat{f}_2) = 0$.

Now note that the set

$$\{\tilde{s} : \tilde{s} = \alpha \cdot s + (1 - \alpha) s', \alpha \in [0, 1], \tilde{s} \cdot (\hat{f}_1 - \hat{f}_2) = 0, s' \in B_{\varepsilon}(\hat{s})\}$$

is convex and its span is all of $\mathbb{R}^{|\{E_i\}|}$. Hence, it has non-empty interior – implying that we can always select an $\hat{s}$ such that it does not lie on any of the (finitely many) planes $\mathcal{L}_i$ defining the partition $\{E_i\}$. Let $E_i$ denote the cell to which $\hat{s}$ belongs and note that we have: $\mathcal{L} \cap \text{int}(E_i) \neq \emptyset$. Since either $\mathcal{L} \cap \text{int}(E_1) \neq \emptyset$ or $\mathcal{L} \cap \text{int}(E_2) \neq \emptyset$, this implies that (letting $f_j$ denote the element of the descriptive representation $M$ chosen in cell $E_j$) we must then have, by lemma 9, either $\text{supp}(f_1) \cap \text{supp}(f_i) = \emptyset$ or $\text{supp}(f_2) \cap \text{supp}(f_i) = \emptyset$. Both possibilities lead to contradiction since $p_1 \in \text{supp}(f_1) \cap \text{supp}(f_2)$ and, since $s^* \cdot (p_1 - p'_1) > 0$, $p_1 \in \text{supp}(f_i)$. The proof of (2) is nearly identical, hence omitted. \qed
Graphical exposition of a special case

We work out the construction when \( k = 3 \) (i.e. three prizes) and \( \{E_i\} \) is a conical partition made up of two hyperplanes. Readers can skip this without loss and continue to Step 3iib of the proof. Let \( \{L_1, L_2\} \) denote the two hyperplanes that generate the partition \( \{E_i\} \), where we put \( L_1 := \{s \in S : (p_1 - p'_1)s = 0\}, L_2 := \{s \in S : (p_2 - p'_2)s = 0\} \) and, by hypothesis, \( p_i \sim_{u, s} p'_i \). Choose coordinates for the cells in the partition formed by \( \{L_1, L_2\} \) as follows,

1. \( E_1 = \{s : (p_1 - p'_1)s \geq 0, (p_2 - p'_2)s \geq 0\} \)
2. \( E_2 = \{s : (p_1 - p'_1)s \geq 0, (p_2 - p'_2)s \leq 0\} \)
3. \( E_3 = \{s : (p_1 - p'_1)s \leq 0, (p_2 - p'_2)s \leq 0\} \)
4. \( E_4 = \{s : (p_1 - p'_1)s \leq 0, (p_2 - p'_2)s \geq 0\} \).

For a schematic representation, when \( k = 3 \) this yields the following partition of the state space:

![Figure 6: A two-hyperplane Borel partition and choice of coordinates E₁ – E₄.](image)

Now add a hyperplane, \( L^* \), to this partition with the following two properties.

1. \( L^* = \{s : (p_s - p'_s)s = 0\} \), where \( u(p_s) = u(p'_s) \).
2. \( \mathcal{L}^* \cap \mathcal{L}_1 = \mathcal{L}^* \cap \mathcal{L}_2 = \mathcal{L}_1 \cap \mathcal{L}_2 \).

That is, the partition of \( S \) induced by these three hyperplanes is conical and, moreover, the state \( s_u \) (where the state-dependent vNM utility is equivalent to the normative ranking on lotteries) lies on the plane \( \mathcal{L}^* \). It is straightforward to see that such a plane \( \mathcal{L}^* \) exists. To see this, let \( W = \mathcal{L}_1 \cap \mathcal{L}_2 \) and let \( s_u \) be the state with cardinal preference given by \( u(\cdot) \). Consider \( V = \text{span}(W, s_u) \) and note that \( V \) has linear dimension \( k - 1 \), hence is itself a hyperplane. Accordingly, we will denote the third hyperplane via \( \mathcal{L}_u \). Denote the augmented partition as \( \{E_i\} \). Take \( s_u \in \text{int}(E_1) \) and note that the plane \( \mathcal{L}_u \) intersects \( E_1 \) and (by the argument in sub-step 3iia) \( E_4 \).

Let \( p_u - p_u' \) denote a normal to \( \mathcal{L}_u \) and coordinatize the cells of the partition formed by \( \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_u\} \):

1. \( \overline{E}_1^1 := \{s : s \in E_1, (p_u - p_u') \cdot s \geq 0\} \)
2. \( \overline{E}_1^2 := \{s : s \in E_1, (p_u - p_u') \cdot s \leq 0\} \)
3. \( \overline{E}_3^1 := \{s : s \in E_4, (p_u - p_u') \cdot s \geq 0\} \)
4. \( \overline{E}_3^2 := \{s : s \in E_4, (p_u - p_u') \cdot s \leq 0\} \)

Figure 7: A three-hyperplane Borel partition with choice of coordinates \( \overline{E}_1^i - \overline{E}_4^i, \overline{E}_2 = E_2, \overline{E}_3 = E_3 \).
Put $M = \{(a_1, a_2, a_3); (f_1, \ldots, f_6)\}$ as a descriptive representation of this partition and note that $\phi_M(s) = \phi_M(s'), \forall s, s' \in E_1$ and, similarly, $\phi_M(s) = \phi_M(s'), \forall s, s' \in E_3$ - so that the $\phi_M$ is measurable w.r.t. to the original partition \{\(E_i\)\}.

I now construct a perturbed menu $\hat{M}$ as in sub-step 3iia and verify that hypothesis (H) is satisfied for this menu. Once we show this, the argument from step 1 shows that $(1, 0, 0, 0)$ (in the $\{E_i\}$ coordinates) is in the span of $\mathcal{U}_\phi$. Find states $s_1, s_2 \in E_1 \cup E_2$ such that $p_u > s_1, p'_u$ and $p_u > s_2, p'_u$. For each $s_i$, find a ball $B_\varepsilon(p_u), B_\varepsilon(p'_u)$ such that (resp.) $p > s_i, p'_u, \forall p \in B_\varepsilon(p_u)$ (and similarly for $p'_u$). Now find $\alpha_1, \alpha_2 > 0$ such that the following two properties are satisfied:

1. $p_u + \alpha_1(p_1 - p'_1) \in B_\varepsilon(p_u), p'_u + \alpha_2(p_2 - p'_2) \in B_\varepsilon(p'_u)$

2. $u(\hat{f}_1) = u(\hat{f}_2)$.

It follows that (H) is satisfied for this choice of $\alpha_i$, implying the span of $\mathcal{U}_\phi$ has dimension 4. From the initial partition we get three basis vectors. In the diagrams below the colored cells represent the support of values of a difference of two Strotzian value functions, each measurable w.r.t. the two-hyperplane partition.

![Figure 8: Two of the basis vectors for $\mathbb{R}^4$ that are in $\text{span}(\Sigma_{E_i})$.](image)

From the schematic we get that $\bar{1}_{E_1 \cup E_3}$, i.e. the indicator function on states in
$E_1 \cup E_3$ and $\bar{I}_{E_2 \cup E_4}$ are in $\text{span}(\Sigma_{E_i})$.

Figure 9: Two more vectors for $\mathbb{R}^4$ that are in $\text{span}(\Sigma_{E_i})$.

We now have four vectors, \{\bar{I}_{E_1 \cup E_3}, \bar{I}_{E_2 \cup E_4}, \bar{I}_{E_3 \cup E_4}, \bar{I}_{E_1 \cup E_2}\}, in $\text{span}(\Sigma_{E_i})$. The problem is that they do not form a basis, e.g. $\bar{I}_{E_1 \cup E_3}$ is linearly dependent on the other three. Hence, we need to find one more (linearly independent) vector in the span. This is provided by the perturbation argument. Notice that since $\mathcal{L}_u$ is an unobservable hyperplane the functions $\phi_M$ and $\hat{\phi}_M$ are both $\Sigma_{E_i}$-measurable.

Figure 10: A fourth basis vector which lies in $\text{span}(\Sigma_{E_i})$.

Note that $\phi_M - \hat{\phi}_M$ is a scalar multiple of $\bar{I}_{E_1}$. By inspection we see that the vectors, $\bar{I}_{E_1 \cup E_2}, \bar{I}_{E_2 \cup E_4}, \bar{I}_{E_3 \cup E_4}, \bar{I}_{E_1}$ form a basis for $\mathbb{R}^4$. 

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Step 3iib: Existence argument for non-conical partitions.

Now we extend to partitions generated by an arbitrary (but finite) number of hyperplanes. Consider a partition \{E_i\} formed by planes \{L_1, \ldots, L_n\}. Label so that \{L_1, \ldots, L_k\} forms a maximal conical partition and such that \(s_u \in \mathcal{C}(L_1, L_2)\). Let \(E_1, E_2, \ldots, E_m\) be an enumeration of cells of the partition, where \(E_i, E_{i+1}\) are adjacent cells in a descriptive representation.\(^{26}\) Note that, by the induction hypothesis, the rows of the following matrix are all in the span of \(U_\phi\).

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix}_{n-1 \times n}
\]

Note that cells \(E_1, E_2\) are separated by a single plane, say \(L_k := \{s : (p_k - p_k') \cdot s = 0\}\). Thus, once we remove this plane the union \(E_1 \cup E_2\) is a cell in the coarsened partition (call it \(E'_1\)) obtained by removing \(L_k\). By the induction hypothesis, the vector \([1, 1, 0, \ldots, 0]\) is in span(\(U_\phi \cap \Sigma_{E'_1}\)) where we identify \(\Sigma_{E'_i}\) with its (embedded) image in \(\Sigma_{E_i}\). Similarly argue to show that the remaining rows of \(A\) are in the span of \(U_\phi\). We now add an unobservable hyperplane to the partition. Arguing as in the base step, add \(L_u\) to the partition, where \(L_u = \text{span}(s_u, L_1 \cap L_2)\) (here we take \(s_u \in \mathcal{C}(L_1, L_2)\)). Let \(E_1, \ldots, E_k\) be a list of the cells that are contained in the cone \(\mathcal{C}(L_1, L_2)\) and let \(f_1, \ldots, f_k\) be the lotteries (resp.) selected by states in \(E_i, \ldots, E_k\) (from the descriptive representation \(M\)), and let \(f_i^u, f_i'^u\) denote the lotteries with small weights attached (resp.) on the pair \((p_u, p'_u)\), where \(p_u - p'_u\) is a normal vector

\(^{26}\)If \(f_i\) is chosen by states in cell \(E_i\) and \(f_{i+1}\) is chosen by states in cell \(E_{i+1}\), then the adjacency condition is that there is exactly one vector \(p_k\) such that \(p_k \in \text{supp}(f_i)\) and \(p_k \in \text{supp}(f_{i+1})\). For all other vectors pairs \((p_j, p'_j)\) we have either \(p_j \in \text{supp}(f_i) \cap \text{supp}(f_{i+1})\) or \(p'_j \in \text{supp}(f_i) \cap \text{supp}(f_{i+1})\).
to the unobservable hyperplane. Consider the following perturbed menu, $\hat{M}$. Put

\[
\hat{f}_i^1 = f_i^1 + \alpha_1^1(p_1 - p'_1), \quad \hat{f}_i^2 = f_i^2 + \alpha_1^2(p_2 - p'_2), \ldots, \quad \hat{f}_k^1 = f_k^1 + \alpha_k^1(p_1 - p'_1), \quad \hat{f}_k^2 = f_k^2 + \alpha_k^2(p_2 - p'_2).
\]

Leave the lotteries $f_i, \forall k + 1 \leq i \leq m$ chosen by states in cells $E_i, \forall i \geq k + 1$ unchanged and put $\hat{M} := \{(\hat{f}_1^i, \hat{f}_2^i)^{k+1})_{i=1}, f_{k+1}, \ldots, f_m\}$.

**Lemma 13.** $\hat{M}$ is $\Sigma_{E_i}$-measurable and one can choose the pairs $(\alpha_1^1, \alpha_1^2)$ such that $\phi_{\hat{M}}$ is constant on the cone $C(L_1, L_2)$.

**Proof of Lemma 13.** The second claim mimics the prior argument verbatim. First, adjust the weights on the descriptive representations so that the differences $\phi_M(s) - \phi_M(s')$ are very small for all $s, s' \in C(L_1, L_2)$. Next, choose $\alpha_1^1, \alpha_1^2$ such that $u(\hat{f}_i^1) = u(\hat{f}_i^2) = u(\hat{f}_j^2), \forall (i, j)$. For the $\Sigma_{E_i}$-measurability claim simply note that if $\hat{f}_i^1$ or $\hat{f}_i^2$ is selected (say $\hat{f}_i^1$) from $\hat{M}$ for any alleged state outside the cone $C(L_1, L_2)$, then we may consider the plane $L := \{s : (\hat{f}_i^1 - \hat{f}_i^2)s \geq 0\}$ and apply the argument (via contradiction) from step 1. Namely, show that (by choice of the constants $\alpha_i$) we have the inclusion of cones $C(L, L_1) \subseteq C(L_1, L_2)$. Having checked $\Sigma_{E_i}$ measurability, consider the difference of the two functions, $\phi_{\hat{M}} - \phi_M$. Wlog say that we have the ordering $\phi_M(s_1) > \phi_M(s_2) > \cdots > \phi_M(s_k), s_i \in \text{int}(E_i)$. We thus obtain $\phi_{\hat{M}} - \phi_M = [0, r_2, r_3, \ldots, r_k, 0, 0, \ldots, 0], r_i < r_{i+1}$. This then implies that

\[
[0, 0, r_3 - r_2, r_4, \ldots, r_k, 0, 0, \ldots, 0] \in \text{span}(\mathcal{U}_\phi)
\]

which in turn implies that $[0, 0, 0, r_4 - (r_3 - r_2), r_5, \ldots, r_k, 0, 0, \ldots, 0] \in \text{span}(\mathcal{U}_\phi)$. Inductively obtain that $[0, 0, \ldots, r_k - \sum_{i=1}^{k-2} (-1)^{i-1}r_{k-i}, 0, 0, \ldots, 0] \in \mathcal{U}_\phi$. It follows that $\text{span}(\mathcal{U}_\phi)$ is full-dimensional. □

**Step 3iii: Bootstrap**

In this step, we patch the measures $\Pi_{E_i}$ across different partitions $\{E_i\}$. This part makes strong use of the duality between the cone in $\mathbb{R}^k$ generated by the set of
differences, \( \{ \phi_M - \phi_{M'} : \phi_M \succeq \phi_{M'} \} \) and the dual cone (also in \( \mathbb{R}^k \) since Euclidean space is self-dual) of (a priori signed) measures on the \( \sigma \)-algebra \( \{ E_i \} \). Since we will need to make frequent reference to these objects, we label them as follows. Put

\[
C_{E_i} := \bigcup_{r > 0} r \mathcal{H}, \quad \mathcal{H} := \{ h \in \mathbb{R}^k : h = \phi_M - \phi_{M'}, M \succeq M' \}
\]

Let \( \mathcal{L}(\mathbb{R}^k) \) be the (vector) space of linear functionals on \( \mathbb{R}^k \) and denote the canonical “duality map” via \( t : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^k) \). Since the set \( C_{E_i} \) has non-empty interior, the self-duality of \( \mathbb{R}^k \) implies that \( C_{E_i}^\prime \) can be uniquely identified with a (closed and convex) cone of measures on the state space \( S \). The following two definitions are standard.

**Definition 7.** Fix a set \( \Lambda \) equipped with a partial order \( \succeq \). A collection of subsets \( \mathcal{C} := \{ C_\alpha \}_{\alpha \in \Lambda} \) is said to be directed downwards if the following holds,

\[
\forall C_\alpha, C_\beta \in \mathcal{C}, \exists \gamma \succeq \alpha, \beta \text{ and surjections } \phi_{\gamma, \beta} : C_\gamma \to C_\beta, \phi_{\gamma, \alpha} : C_\gamma \to C_\alpha
\]

**Definition 8.** Fix a downwards directed system \( \mathcal{C} = \{ C_\alpha \}_{\alpha \in \Lambda} \), where \( C_\alpha \subseteq \Delta(X) \). A subset of the cartesian product \( \prod_{\lambda \in \Lambda} \Delta(S) \) is called an inverse limit of the system \( \mathcal{C} \), denoted \( \varprojlim C_\alpha \), if it is the smallest subset with the property that there is a surjective map \( \phi_\alpha : \varprojlim C_\alpha \to C_\alpha, \forall \alpha \in \Lambda \) that is compatible with the individual surjections \( \phi_{\gamma, \beta} \).

In the latter definition, compatibility means that

\[
\phi_\beta(x) = \phi_{\gamma, \beta}(\phi_\gamma(x)),
\]

when \( \gamma \succeq \beta \). Let \( \Lambda \) be the collection of all (coherent) Borel partitions of \( S \) and take the relation \( \succeq \) to be \( \{ E_i \} \succeq \{ F_i \} \) iff \( \{ E_i \} \) is a refinement of \( \{ F_i \} \). This is a well-defined partial order. Consider the collection, \( \{ \Sigma_{E_i} \} \), of \( E_i \)-measurable menus and let \( \mathcal{C} := \{ C_{E_i} \} \) – where \( C_{E_i} \) is the positive cone of \( \{ E_i \} \) measurable functions. Let \( C_{E_i}^\prime \) denote the set of measures on the Borel partition \( \{ E_i \} \), obtained from the finite subjective Bewley theorem.
Lemma 14. The system \( \{C_{E_i}^t\} \) is downwards directed.

Proof of Lemma 14. Notice that if \( \{F_i\} \supseteq \{E_i\} \), then the set \( \Sigma_{E_i} \subseteq \Sigma_F \) and is closed. Thus, \( C_{E_i} \subseteq C_F \) (as a closed subset). Let \( \iota(C_{E_i}) \) denote the image of \( C_{E_i} \) in \( C_F \). For each \( \pi \in C_{E_i}^t \) we put

\[
\text{proj}(\pi)(D) := \sum_{F_i : F_i \cap D \neq \emptyset} \pi(F_i)
\]

where \( D \in \{E_i\} \). Note that \( \text{proj}(\pi) \) is a well-defined measure on \( \{E_i\} \). Let \( x, y \) denote generic elements of \( \Sigma_{E_i} \) and note that we have the following sequence of implications,

\[
x \succeq y \iff \iota(x) \succeq \iota(y)
\]

\[
\iff \int_{\{F_i\}} \iota(x) \, d\pi \geq \int_{\{F_i\}} \iota(y) \, d\pi, \, \forall \pi \in C_{E_i}^t
\]

\[
\iff \int_{\{E_i\}} x \, d\text{proj}(\pi) \geq \int_{\{E_i\}} y \, d\text{proj}(\pi), \, \forall \pi \in C_{F_i}^t.
\]

Since this holds for all pairs \( x, y \in C_{E_i} \), by the uniqueness of the Bewley representation we obtain that the projection map, \( \text{proj} : C_{F_i}^t \to C_{E_i}^t \), is a surjection. \( \square \)

Uniqueness of the Bewley representation for \( \Sigma_{E_i} \)-measurable acts is standard, but here we want to prove the same result for \( \Sigma_{E_i} \)-measurable menus. The proof from the Anscombe-Aumann setting does not apply in this case. First consider the following preliminary step. In the classical (Anscombe-Aumann) setting the lemma is obvious since we have (i) the monotonicity axiom and (ii) a richer collection of choice objects (namely, acts as opposed to menus), which immediately implies that the cone \( C_{E_i}^t \) consists of positive measures. When these measures are, a priori, signed and the domain of choice is menus, the following claim then requires a proof.

Lemma 15. \( C_{E_i}^t = \{d\pi : d \geq 0, \pi \in C_{E_i}^t \text{ s.t. } \sum_i \pi_i = 1\} \).
Proof of Lemma 15. For any \( \pi \in C^t_{E} \) we consider the sum of the coordinates, \( \sum_i \pi_i \), of the measure \( \pi \). Take two singleton menus, \( M_1 = \{ \ell_1 \}, M_2 = \{ \ell_2 \} \) with \( u(\ell_1) > u(\ell_2) \). Then, since \( E_\pi \phi_{M_1} \geq E_\pi \phi_{M_2} \), we must have \( \sum_i \pi_i \geq 0 \). We claim the inequality is strict. Else, consider a \( \pi \) with \( \sum_i \pi_i = 0 \). Since \( \Sigma_{E_i} \) has non-empty interior, the cone \( C_{E_i} \) has non-empty interior. This implies that a linear function vanishes on this cone if and only if it is the zero function. Thus, there is some \( \{ E_i \} \)-measurable menu \( M \) with \( E_\pi \phi_M > 0 \) (strict positivity since we must have \( M \succ \{ \ell \} \), where \( \ell \) is a \( u \)-minimal lottery). Consider mixtures \( M\alpha \{ \bar{\ell} \} \), where \( \bar{\ell} \) is a \( u \)-maximal lottery. Since \( M\alpha \{ \bar{\ell} \} \succeq M \) we have: \( \phi_{M\alpha \{ \bar{\ell} \}} - \phi_M \in C_{E_i} \). It follows that

\[
E_\pi \phi_{M\alpha \{ \bar{\ell} \}} \geq E_\pi \phi_M, \forall \alpha \in [0, 1].
\]

On the other hand, we also have (since \( \sum_i \pi_i = 0 \))

\[
E_\pi \phi_{M\alpha \{ \bar{\ell} \}} = E_\pi [\alpha \phi_M + (1 - \alpha)\phi_{\{ \ell \}}] = \alpha E_\pi \phi_M.
\]

Hence, \( \alpha E_\pi \phi_M \geq E_\pi \phi_M, \forall \alpha \) – contradiction. \( \square \)

For any Bewley representation, it suffices to take any (closed, convex) generating set for the dual cone \( C^t_{E_i} \). By the preceding lemma, we will take \( \Pi \) (the generating set of measures) to be those measures whose coordinates sum to one.

Proposition 3. Let \((u, \Pi_1), (u, \Pi_2)\) denote a pair of Bewley representations of \( \succeq \subseteq \Sigma_{E_i} \times \Sigma_{E_i} \). Then, \( \Pi_1 = \Pi_2 \).\(^{27}\)

First, a preliminary step. Let \( C \subseteq \mathbb{R}^n \) be any closed and convex cone, and assume we have two closed, convex sets of functionals in \( \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \), say \( \mathcal{L}_1, \mathcal{L}_2 \), such that all elements of \( \mathcal{L}_1, \mathcal{L}_2 \) are represented by dot products with measures whose coordinates sum to one.

\(^{27}\)For this proposition (only), we say that a Bewley representation of \( \succeq \) is one in which \( f \succeq g \iff E_\pi u(f) \geq E_\pi u(g), \forall \pi \in \Pi \) and where the set \( \Pi \) is closed and consists of measures (i.e. not necessarily positive). The proposition shows that uniqueness (hence, our bootstrap argument) holds for this broader class of models.
Lemma 16. If $C = \cap_{\ell \in \mathcal{L}_1} \{ x : \ell(x) \geq 0 \} = \cap_{\ell \in \mathcal{L}_2} \{ x : \ell(x) \geq 0 \}$, then $\mathcal{L}_1 = \mathcal{L}_2$.

Proof of Lemma 16. Towards contradiction, say that $\mathcal{L}_1 \neq \mathcal{L}_2$. If $\mathcal{L}_1 \nsubseteq \mathcal{L}_2$ take $\ell_1 \in \mathcal{L}_1 \setminus \mathcal{L}_2$. Note that $\{ \ell_1 \}$ and $\mathcal{L}_2$ are closed, convex, and disjoint sets in $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ (where $\{ \ell_1 \}$ is obviously bounded). By the Separating Hyperplane Theorem, there is a functional $\hat{x} \in \mathcal{L}(\mathcal{L}(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$ such that $\hat{x}(\ell_1) < c < \hat{x}(\ell_2), \forall \ell_2 \in \mathcal{L}_2$. Since $\mathbb{R}^n$ is reflexive, the functional $\hat{x}$ is given by point evaluation for some $x \in \mathbb{R}^n$. That is, $\hat{x}(\ell) = \ell(x), \forall \ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ (for some $x \in \mathbb{R}^n$). Since the functionals in $\mathcal{L}_1, \mathcal{L}_2$ are all represented by dot products with measures whose coordinates sum to one, consider $x' := x - c \cdot \overline{1}$ and note that $\ell(x') = \ell(x) - c, \forall \ell \in \mathcal{L}_1, \mathcal{L}_2$. It follows that

$$\ell_1(x') < 0 < \ell_2(x'), \forall \ell_2 \in \mathcal{L}_2.$$ 

Therefore, since $C = \cap_{\ell \in \mathcal{L}_2} \{ x : \ell(x) \geq 0 \}$, we obtain $x' \in C$. OTOH, we also have $C = \cap_{\ell \in \mathcal{L}_1} \{ x : \ell(x) \geq 0 \}$. Thus, if $x' \in C$ then $x' \in \{ x : \ell_1(x) \geq 0 \}$ - contradiction. Thus, if $\mathcal{L}_1 \neq \mathcal{L}_2$ then we must have $\mathcal{L}_2 \nsubseteq \mathcal{L}_1$. The symmetric argument taking $\{ \ell_2 \}$ and $\mathcal{L}_1$ to be the separated sets of functionals, shows that $\mathcal{L}_2 \nsubseteq \mathcal{L}_1$ is also not possible. It follows that $\mathcal{L}_1 = \mathcal{L}_2$. \hfill \qed

We will now use this fact to show the uniqueness claim of the proposition.

Proof of Proposition 3. For this step, we consider a translation of the set, $\Sigma_{E_i}$, of translated $\{ E_i \}$-measurable Strotzian value functions, i.e. put $u := \min_{x \in \Delta(X)} u(x)$ and translate every value function $\phi_M(s) \in \mathbb{R}^n$ by the vector $\underline{u} := (u, \ldots, u)$. Note that (i) the translated set $\Sigma_{E_i} - \underline{u}$ inherits the affine (partial) order from $\Sigma_{E_i}$ and (ii) the translation washes out when we look at the pointed cone, i.e. the pointed cone in the original utility space is identical to the pointed cone in the translated space. Hence, the representation result is unchanged. Thus, for the remainder of the argument I work with the translated sets $\Sigma_{E_i} - \underline{u}$ and will denote generic elements of this set by $f,g,$ etc. A virtue of this translation is that the origin is now an element of both $\Sigma_{E_i} - \underline{u}$ and the pointed cone. Moreover, since $\Sigma_{E_i}$ has non-empty
interior in $\mathbb{R}^n$ we also have that $\Sigma_{E_i} - \vec{u}$ has non-empty interior in $\mathbb{R}^n$. Hence, span($\Sigma_{E_i} - \vec{u}$) = $\mathbb{R}^n$. Now let us say that we have two sets of measures $\Pi_1, \Pi_2$ that both yield a Bewley representation of $\succeq$, and let us take $\Pi_1$ to be the set obtained as the dual to the pointed cone. That is, we put

$$C_{E_i} = \cap_{\ell \in \Pi_1} \{ x \in \mathbb{R}^n : \ell(x) \geq 0 \}$$

Now if $\Pi_2$ also yields a Bewley representation, then we must have

$$C_{E_i} \subseteq \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}$$

Hence, $\Pi_2 \subseteq \Pi_1$ (as taking duals reverses the inclusion and produces $\Pi_2$ on the RHS and $\Pi_1$ on the LHS). Note that if $C_{E_i} = \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}$, then the preceding lemma yields equality of the measures. Hence, consider the possibility that $C_{E_i} \subseteq \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}$. Take $z \in \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}\setminus C_{E_i}$ and write $z = \sum_{i=1}^{M} \alpha_i f_i - \sum_{j=1}^{N} \beta_j g_j$ where $f_i, g_j \in \Sigma_{E_i} - \vec{u}$ and $\alpha_i, \beta_j \geq 0$ (since span($\Sigma_{E_i} - \vec{u}$) = $\mathbb{R}^n$). Put

$$\hat{f} = \frac{1}{\sum_i \alpha_i} \sum_i \alpha_i f_i, \quad \hat{g} = \frac{1}{\sum_j \beta_j} \sum_j \beta_j g_j$$

and set $A = \sum_i \alpha_i, \quad B = \sum_j \beta_j$ (note that $B > 0$ as $z \notin C_{E_i}$). Now notice that $z = A\hat{f} - B\hat{g}$. Divide by max($A, B$) to get

$$\hat{z} := \frac{z}{\max(A, B)} = \frac{A}{\max(A, B)} \hat{f} - \frac{B}{\max(A, B)} \hat{g}$$

Observe that $\hat{z} \in \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}\setminus C_{E_i}$ as both sets are cones. Moreover, since $\hat{f}, \hat{g}, \vec{0} \in \Sigma_{E_i} - \vec{u}$ we have $\frac{A}{\max(A, B)} \hat{f}, \frac{B}{\max(A, B)} \hat{g} \in \Sigma_{E_i} - \vec{u}$. Hence, $\hat{z}$ is a difference of two elements of $\Sigma_{E_i} - \vec{u}$, so that we put $\hat{z} = x - y$. Since $\hat{z} \notin C_{E_i}$ and $\Pi_2$ provides a Bewley representation for $\succeq$ it follows that there is some $\ell \in \Pi_2$ such that $\ell(\hat{z}) < 0$ – contradicting the hypothesis that $\hat{z} \in \cap_{\ell \in \Pi_2} \{ x : \ell(x) \geq 0 \}$. Hence, $\Pi_1 = \Pi_2$. \hfill $\square$

Returning to the main argument, it follows that the system, $C := \{ C_{E_i} \}$ corresponds, via the Bewley representation, to the downwards directed system, $C^\prime := \{ C_{E_i} \}$.
\{C_{E_i}^t\}. Upon taking inverse limits, this system yields the set of (finitely additive) measures for the Bewley representation. For this, we just check that the inverse limit exists and is closed. Let \(X := \prod_{\{E_i\}} \Delta(\{E_i\})\) and endow this space with the product topology (recall: the topology on \(ca(S)\) is the topology of weak convergence). Consider the set \(\overline{X} \subseteq X\) defined by \(\overline{X} := \prod_{\{E_i\}} C_{E_i}^t\). Let \(\overline{\lim C}^t\) be defined as follows. Since there are uncountably many (coherent) Borel partitions \(\{E_i\}\) and this collection is endowed with a natural partial order \(\succeq^*\), let us index the collection with the set \(\Lambda\). Put

\[
\overline{\lim C}^t := \{\pi = (\pi_\alpha, \pi_\beta, \ldots) : \gamma \succeq^* \alpha, \beta \Rightarrow \text{proj}_{\gamma,\alpha}(\pi_\gamma) = \pi_\alpha, \text{proj}_{\gamma,\beta}(\pi_\gamma) = \pi_\beta\}
\]

Notice that this is clearly a closed subset of \(X\). Moreover, notice that it can naturally be identified with a subset of the set of measures on \(\mathcal{A}\). To see this, let \(\mathcal{B}\) denote the \(\sigma\)-algebra on \(S\) generated by all (finite) Borel partitions of \(S\) and recall that \(\mathcal{A} = \bigcup_{\alpha \in \Lambda} \sigma(\{E_i\})\) is the algebra generated by all finite Borel partitions. Notice that every element of \(\overline{\lim C}^t\) determines (uniquely) a (finitely-additive) measure on the algebra \(\mathcal{A}\). Similarly, any measure on \(\mathcal{A}\) determines an element of the set \(\overline{\lim C}^t\).

In other words, as sets we can identify \(\overline{\lim C}^t\) with the collection of measures on \(\mathcal{A}\). Since the product topology on the set \(\overline{\lim C}^t\) coincides with the topology of weak convergence\(^{28}\) when we identify this set with measures on \(\mathcal{A}\), the identification is a topological isomorphism.

This formally concludes the bootstrap step. The preceding arguments constructed measures on the algebras generated by finite partitions \(\{E_i\}\). Before proceeding to the next step, we require a result on convergence of these finitely-additive measures along “shrinking” sequences of cells. More precisely, we show that the value of \(\pi\) (for any \(\pi \in \Pi\)) on cones \(E_n\) of the form

\[
E_n(p_n, q) := \{s : (p_n - p_n') \cdot s \geq 0, (q - q') \cdot s < 0\}
\]

\(^{28}\)Here we take the class of test functions to be the set of \(\mathcal{A}\)-measurable functions.
is vanishing. Note that \( \cap_n E_n(p_n, q) = \emptyset \). The following proposition shows that vanishing weight, for any \( \pi \), is placed on the \( E_n(p_n, q) \) for all large \( n \).

**Proposition 4** (Convergence on Shrinking Cells). Let \( p_n \to q \) and define the cones 
\[
E_n(p_n, q) := \{ s : (p_n - p'_n) \cdot s \geq 0, (q - q') \cdot s < 0 \}.
\]
Then, \( \pi(E_n(p_n, q)) \to 0 \) as \( p_n \to q \).

**Proof of Proposition 4.** This fact will also be applied in a forthcoming argument, where we will require the measures to vanish on nested cells in what we call a “\( k \)-wise conical” partition, i.e. where cells are cones supported by more than two planes. Generalizing to the \( k \)-wise conical case adds a layer of notation to the result (see footnote 34 for the extension). Hence, we restrict for now to (pairwise) conical cells and make a remark on how one does the extension when the need arises. We abuse notation and use \( p^n \) for perturbed lotteries which converge to \( q \). These lotteries will then be use to construct perturbed descriptive representations with associated \( f^n \).

The sets \( E_n \) in question will be defined in terms of the lotteries \( f^n \). We show, after the claim below, that moving between the \( p_n \) and associated \( f^n \) in the manner described is a two-way street. Let \( \{ E_i \} \) denote a conical partition and let \( \{ p_1 - p'_1, \ldots, p_k - p'_k \} \) enumerate the (normals to) planes defining the partition. Selecting a \( q - q' := p_i - p'_i \) in this set we let \( p^n = q + \varepsilon_n \cdot (p_{i+1} - p'_{i+1}) \to q \). Let \( M \) be a descriptive representation of \( \{ E_i \} \) and put \( E_i := \{ s : (p_i - p'_i) \cdot s < 0, (p_{i+1} - p'_{i+1}) \cdot s \geq 0 \} \). Let \( f_j \) denote the lottery from \( M \) chosen in state \( E_j \). Recall that we have \( f_j = \sum_{i=1}^k \alpha_i \cdot p_j(l) \), where \( p_j(l) \in \{ p_i, p'_i \} \). Now consider the augmented menu \( \hat{M}^n := M \cup \{ \hat{f}_i^n \} \), where we put
\[
\hat{f}_i^n = \sum_{l \neq i} \alpha_i \cdot p_i(l) + \frac{\alpha_i}{2} \cdot p_i(l) + \frac{\alpha_i}{2} \cdot p^n.
\]
We need a lemma which mirrors the arguments for (resp.) lemmas 9-12 and proposition 2. Define \( \mathcal{L}_{i+1} := \{ s : (p_{i+1} - p'_i) \geq 0 \} \), \( \mathcal{L}^n := \{ s : (f_i^n - f_i) = 0 \} \) and let \( \mathcal{C}(\mathcal{L}^n, \mathcal{L}_{i+1}) := \{ s : (i) \; s \cdot (f_i^n - f_i) \geq 0, (ii) \; s \cdot (p_{i+1} - p'_i) \geq 0 \} \). “IC” is short for incentive compatibility.

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Lemma 17 (IC, revisited). \( \mathcal{C}(\mathcal{L}^n, \mathcal{L}_{i+1}) \subseteq E_{i-1} \cup E_i \).\(^{29}\)

Proof of Lemma 17. The argument follows along similar lines to those in lemmas 9-12, and proposition 2. We first check that starting out with a conical partition, the augmented partition obtained by adding \( \mathcal{L}^n \) is also conical. Second, we check that if \( \mathcal{L}^n \cap E_j, \mathcal{L}^n \cap E_k \), then \( \text{supp}(f_j) \cap \text{supp}(f_k) = \emptyset \). Third, analogous to proposition 2, we show that these two facts imply incentive compatibility. Let us first check that the augmented partition is conical. Notice that \( f_{i-1}^n - f_i = \alpha_i \cdot (q - q') + \varepsilon_n \frac{\alpha_i}{2} (p_{i+1} - p'_{i+1}) \). Hence, \( \mathcal{L}_q \cap \mathcal{L}^n \subseteq \mathcal{L}_q \cap \mathcal{L}_{i+1} \). Counting linear dimensions gives equality, so that \( \mathcal{L}_q \cap \mathcal{L}^n = \mathcal{L}_q \cap \mathcal{L}_{i+1} \). Similarly, we find \( \mathcal{L}^n \cap \mathcal{L}_{i+1} = \mathcal{L}_q \cap \mathcal{L}_{i+1} \). Now notice that: \( \mathcal{L}^n \cap \mathcal{L}_j \cap \mathcal{L}_q \subseteq \mathcal{L}^n \cap \mathcal{L}_j \) for any \( \mathcal{L}_j \). On the other hand, \( \mathcal{L}^n \cap \mathcal{L}_q = \mathcal{L}_q \cap \mathcal{L}_{i+1} \). Since the original partition is conical we find, \( \mathcal{L}_q \cap \mathcal{L}_j \cap \mathcal{L}_q = \mathcal{L}_j \cap \mathcal{L}_q \cap \mathcal{L}_j \). Counting dimensions we obtain \( \mathcal{L}_j \cap \mathcal{L}_k = \mathcal{L}^n \cap \mathcal{L}_j \). Since \( j \) is arbitrary, we obtain that the augmented partition is conical. A verbatim replica of the argument in lemma 12 now shows that if \( \mathcal{L}^n \cap \text{int}(E_j), \mathcal{L}^n \cap \text{int}(E_k) \neq \emptyset \), then \( \text{supp}(f_j) \cap \text{supp}(f_k) = \emptyset \). Now we verify the IC condition.\(^{30}\) Towards contradiction, say that \( \mathcal{C}(\mathcal{L}^n, \mathcal{L}_{i+1}) \nsubseteq E_{i-1} \cup E_i \) and (wlog) let \( s \in \mathcal{C}(\mathcal{L}^n, \mathcal{L}_{i+1}) \setminus (E_{i-1} \cup E_i) \) be such that \( s \cdot (f_{i-1}^n - f_i) > 0 \) (and, by hypothesis, \( s \cdot (p_{i-1} - p'_{i-1}) < 0 \)). Find another state \( \hat{s} \) (\( \neq E_{i-1} \cup E_i \)) such that:

i. \( \hat{s} \cdot (p_{i+1} - p'_{i+1}) > 0 \),

ii. \( \hat{s} \cdot (f_{i-1}^n - f_i) < 0 \), and

iii. \( \hat{s} \cdot (p_{i-1} - p'_{i-1}) < 0 \).

Find \( \alpha \in (0, 1) \) such that \( (\alpha \cdot s + (1 - \alpha) \hat{s}) \cdot (f_{i-1}^n - f_i) = 0 \) and let \( s^* := \alpha \cdot s + (1 - \alpha) \cdot \hat{s} \).

Note that \( s^* \in \mathcal{L}^n \) and \( s^* \notin E_{i-1} \cup E_i \) (since \( s^* \cdot (p_{i-1} - p'_{i-1}) < 0 \)). Hence, there is

\(^{29}\)Since the original partition is conical note that we can express, \( E_{i-1} \cup E_i = \{ s : (i) s \cdot (p_{i-1} - p'_{i-1}) \geq 0, (ii) s \cdot (p_{i+1} - p'_{i+1}) \geq 0 \} \).

\(^{30}\)This argument mimics proposition 1. However, the premises are slightly different so we do the proof in the interest of completeness.
some cell $E_j$ with $s \in \mathcal{L}^n \cap E_j$. Since the inequalities defining $\hat{s}$ are strict, exactly as in
the original IC argument (proposition 1) we can assume wlog that $s \in \mathcal{L}^n \cap \text{int}(E_j)$.
This then implies that $\text{supp}(f_j)$ (the lottery chosen in cell $E_j$) has null intersection
with either $f_{i-1}$ or $f_i$. However, by construction, $p_{i+1} \in \text{supp}(f_{i-1}) \cap \text{supp}(f_i)$ and
$s^*(p_{i+1} - p'_{i+1}) > 0$, which implies that $p_{i+1} \in \text{supp}(f_k)$ (the lottery chosen by stets
in $E_k$) – contradiction. Hence, IC holds.

Return now to the proof of the proposition. Start with any sequence $p_n \to q$
and show that there is an associated $f^n_{i-1}$ such that $E_n(p_n, q) = \{ s : s\cdot(p_n - p_n') \geq 0, s\cdot(q - q') < 0 \} = \{ s : s\cdot(f^n_{i-1} - f_i) \geq 0, s\cdot(q - q') < 0 \}$. By lemma 15 we have
that $p_n = \gamma_n q + (1 - \gamma_n)p_{i+1}$ for some sequence $\gamma_n \in (0, 1), \gamma_n \uparrow 1$. Note that
$p_n - p'_n = \gamma_n(q - q') + (1 - \gamma_n)(p_{i+1} - p'_{i+1})$. Given $\gamma_n$ and weights $\alpha_l$ (which wash
out) we find the unique $\varepsilon_n$ which solves:

$$\gamma_n = \frac{\alpha_l}{\alpha_l + \varepsilon_n \alpha_l / 2}$$

For this $\varepsilon_n$ we consider $p^n$ as above and, for the associated perturbed lotteries
$f^n_{i-1}$ we find that $\{ s : s\cdot(f^n_{i-1} - f_i) \geq 0, s\cdot(q - q') < 0 \} = E_n(p_n, q)$. Proceed now
with the main argument. Consider the functions $\phi_{M_n}$ and note that, by the claim,
we have $\phi_{\hat{M}_n}(s) = \phi_M(s), \forall s \notin E_{i-1} \cup E_i$. Let $\pi \in \Pi$ and compute the quantity
$$\lim_n E_\pi(\phi_{\hat{M}_n} - \phi_M).$$
We obtain:

$$\lim_n E_\pi(\phi_{\hat{M}_n} - \phi_M) = \lim_n \pi(E_n(p_n, q)) \cdot (u(f^n_{i-1}) - u(f_i))$$
$$= \lim_n \pi(E_n(p_n, q)) \cdot (\alpha_l(u(q) - u(q')) + \frac{\alpha_l \varepsilon_n u(p_{i+1}) - u(p'_{i+1})}{\beta_n}).$$

Notice that $u(f^n_{i-1}) - u(f_{i-1}) \to 0$, so we omit the (vanishing) utility change on
the cell $E_{i-1}$ for finite $n$. Also, observe that (i) the plane $\mathcal{L} := \{ s : s\cdot(f^n_{i-1} - f_i) = 0 \}$ intersects the interior of cell $E_i$ and (ii) if $\pi(E_n(p_n, q)) \not\to 0$, then the limit is
non-zero (since $u(q) - u(q') \neq 0$, so that $\lim \beta_n = \beta \neq 0$). From this, we will
derive a contradiction to axiom 3 (continuity). Hence, towards this contradiction,
assume that \( \lim_n \pi(E_n(p_n, q)) \neq 0 \). Consider the case where the limit is positive (the argument in the negative case is symmetric). Notice that, since \( \cap_n E_n(p_n, q) = \emptyset \), the sequence \( \widehat{M}^n \to M \) and is regular (since \( \phi_{\widehat{M}^n} \to \phi_M \), this implies regularity – see proposition 6).\(^{31}\) We will need the fact that the expected utility difference, \( E_\pi \phi_{\widehat{M}^n} - E_\pi \phi_M \), limits to \( (\lim_n \pi(E_n(p_n, q))) \cdot \beta \neq 0 \). Restricting the representation to \( \Sigma_{E_i} \), we may wlog consider the case where there is some pair of \( \{E_i\}\)-measurable menus \( (M, M') \) for which:

1. \( E_{n'} \phi_M \geq E_{n'} \phi_{M'}, \forall \pi' (\neq \pi) \in \Pi \).
2. \( E_{n'} \phi_M < E_{n'} \phi_{M'} \).

That is, the measure \( \pi \) “justifies” the incomparability between the pair \( (M, M') \).

By mixing both \( M, M' \) with (resp.) a common weight on the menu \( \{p_i, p'_i\} \) (resp. \( \{p_{i+1}, p'_{i+1}\} \)) we can assure that (i) the borders \( p_i - p'_i, p_{i+1} - p'_{i+1} \) are visible in both menus \( M, M' \) and (ii) the weight \( \alpha_i \) (described above) in both menus is non-zero. Now consider the construction \( \widehat{M}^n \) (applied to the given menu \( M \)). By the lemma, the only states in which a difference in second stage choice between \( \widehat{M}^n \) and \( M \) occurs is for states in the cell bordered by \( p_i - p'_i, p_{i+1} - p'_{i+1} \). Moreover, utility strictly increases in those states where the switch occurs. Since the measures in \( \Pi \) are all finitely additive and positive, it follows that \( E_{n'} \phi_{\widehat{M}^n} \geq E_{n'} \phi_M, \forall \pi' \in \Pi \). Replace \( M \) with \( \alpha \cdot M + (1 - \alpha) \cdot M' \) for \( \alpha \) close to 0. Since the weights \( \alpha_i \) in menus \( M \) and \( M' \) are both non-negligible. The weight in the mixture (for any \( \alpha \)) is non-negligible. Note that, by abuse, calling the mixture \( M \), the above inequalities hold for the pair \( (M, M') \) as well. We claim we can pick \( \alpha \) small so that (for the \( \widehat{M}^n \) associated to the mixture \( M \))

\[
(**) \ E_\pi \phi_{\widehat{M}^n} > E_\pi \phi_{M'}, \forall n.
\]

\(^{31}\)If the menu preference \( \geq \) were complete (i.e. there was just a single prior), then we would take \( u \) such that \( E_\pi \phi_{\widehat{M}^n} \geq u, \forall n \) large, where we choose \( u \) such that \( |E_\pi \phi_{\widehat{M}^n} - u| < \pi(E_n(p_n, q)) \cdot \beta \). Taking limits and applying continuity we would need, on the one hand, \( E_\pi \phi_M \geq u \). On the other, by choice of \( u \), we have \( E_\pi \phi_M < u \) – contradiction.
To see this, note that for \( \alpha \) small we have \( E_\alpha \phi_M - E_\alpha \phi_{M'} \to 0 \). However, the weight \( \alpha \) on the lotteries \( p_i, p'_i \) in the menu \( M \) is non-vanishing in \( \alpha \). Hence, since
\[
\lim_n E_\alpha \phi_{\hat{M}^n} - E_\alpha \phi_M \to (\lim_n \pi(E_n(p_n, q))) \cdot \beta > 0,
\]
we can take a small \( \alpha \) such that \((**)\) holds. This implies that \( E_\alpha \phi_{\hat{M}^n} \geq E_\alpha \phi_{M'} \), \( \forall \pi' \in \Pi \), so that \( \hat{M}^n \succeq M' \). By axiom 3, this implies \( M \succeq M' \) (since \( \hat{M}^n \to M \) regularly) – contradiction. \( \square \)

**Step 3iv: Positivity of measures.**

Standard arguments now imply that, for each fixed (coherent) Borel partition \( \{E_i\} \), the cone in \( \mathbb{R}^{\{|E_i|\}} \),

\[
C_{E_i} := \{ \phi_M - \phi_{M'} : M, M' \in \Sigma_{E_i}, M \succeq M' \}
\]
is closed. Hence, the cone equals the intersection of all half-spaces which contain it. Put \( k := |\{E_i\}| \) and let \( \Pi_{E_i} \) denote the (closed) collection measures on \( \{1, 2, \ldots, k\} \) which are normals to these supporting hyperplanes. We now verify that these measures are all positive (i.e. not signed), so that we may take (after normalization) \( \Pi_{E_i} \) to be probability measures. Enumerate the cells of the partition \( \{E_i\} \) as \( \{E_1, \ldots, E_k\} \), where the labeling is chosen so that \( s_n \in E_1 \) (i.e. the normative state lies in \( E_1 \)). We first require some lemmas.

Let \( \{p_1 - p'_1, p_2 - p'_2, \ldots, p_k - p'_k\} \) denote the hyperplanes defining a conical partition. Let \( p_i - p'_i, p_{i+1} - p'_{i+1} \) be the hyperplanes supporting the cone \( E_i \), where we label such that \( E_i = \{ s : (p_i - p'_i) \cdot s < 0, (p_{i+1} - p'_{i+1}) \cdot s \geq 0 \} \). Add an (observable) hyperplane to this partition, viz. take any state \( s_{E_i} \in \text{int}(E_i) \) and consider the hyperplane \( sp(\mathcal{L}_i \cap \mathcal{L}_j, s_{E_i}) \) where \( \mathcal{L}_j = \{ s : (p_j - p'_j) \cdot s = 0 \} \). As per previous arguments, the linear dimension of the space \( sp(\mathcal{L}_i \cap \mathcal{L}_j, s_{E_i}) \) is \( k - 1 \), implying it is a hyperplane. After normalizing we let \( q - q' \) be a normal with \( q \) a lottery. Let \( \hat{E}_j \) denote the augmented partition and label so that \( \hat{E}_i \cup \hat{E}_{i+1} = E_i \).

**Lemma 18.** The augmented partition \( \{ \hat{E}_i \} \) is conical.
Proof of Lemma 18. To see this, let $L_q = \text{sp}(L_i \cap L_{i+1}, s_{E_i}) = \{ s : (q - q') \cdot s = 0 \}$ and consider $L_q \cap L_i$. Writing $s \in L_q$ as $s = \sum_j a_j s_l + b s_{E_i}, s_l \in L_i \cap L_{i+1}$ we obtain that if $s \in L_i$, then $b = 0$ as $s_{E_i} \in \text{int}(E_i)$. Hence, $s \in L_i \cap L_{i+1}$, implying that $L_q \cap L_i \subseteq L_i \cap L_{i+1}$. The reverse containment is obvious, so that $L_q \cap L_i = L_i \cap L_{i+1}$. Similarly, consider any $L_q \cap L_j$. Since the original partition is conical we have $L_j \cap L_i = L_i \cap L_{i+1}$. Hence, if $s = \sum_l a_l s_l + b s_{E_i}, s_l \in L_i \cap L_{i+1}$ and $s \in L_j$, then $b = 0$, implying that $L_q \cap L_j \subseteq L_i \cap L_{i+1}$. Since the reverse containment holds as well (by definition of $L_q$ and since the original partition is conical), we obtain $L_q \cap L_j = L_i \cap L_{i+1} = L_q \cap L_i, \forall j$ – so that the augmented partition is also conical.

Recall that in defining $\{\bar{E}_i\}$ we simply took any state $s \in \text{int}(E_i)$ and put $L = \text{sp}(s_{E_i}, L_i \cap L_{i+1})$. Let $q - q'$ be the normal to $L$ and (resp.) $p_i - p_i', p_{i+1} - p_{i+1}'$ denote the normals to $L_i$ (resp. $L_{i+1}$).

**Lemma 19.** The set of $q - q'$ which are normals to planes of the form $L := \text{sp}(s, L_i \cap L_{i+1})$ can wlog be taken to be $q = \gamma \cdot p_i + (1 - \gamma) \cdot p_{i+1}$.

The lemma associates a state lying in the cone $C(L_i, L_{i+1})$ to a tradeoff rate, viz. $\gamma/(1 - \gamma)$, for which a DM with cardinal consumption utility $u_s := s$ “trades” units of the commodity, $p_i - p_i'$, against units of $p_{i+1} - p_{i+1}'$.

**Proof of Lemma 19.** Note that the conclusion is equivalent to claiming that $q' = \gamma \cdot p_i' + (1 - \gamma) \cdot p_{i+1}'$. To show this, first put $q = \gamma \cdot p_i + (1 - \gamma) \cdot p_{i+1}$. We find an $s_q \in E_i$ such that $q - q'$ is normal to $L = \text{sp}(s_q, L_i \cap L_{i+1})$. Note that $q - q' = \gamma \cdot (p_i - p_i') + (1 - \gamma) \cdot (p_{i+1} - p_{i+1}')$. For every $s \in \text{int}(E_i)$ we have $s \cdot (p_i - p_i') < 0, s \cdot (p_{i+1} - p_{i+1}') > 0$. Clearly then, for each such $s$ there is a $\gamma_s$ such that, putting $q_s := \gamma_s \cdot p_i + (1 - \gamma_s) \cdot p_{i+1}$, we obtain $s \cdot (q_s - q'_s) = 0$. As we take a sequence $s_n \in \text{int}(E_i)$ with $s_n \rightarrow L_{i+1} \setminus L_i$ we obtain $s_n \cdot (p_{i+1} - p_{i+1}') \rightarrow 0$. Hence, we must have $(1 - \gamma_{s_n}) \uparrow 1$. Similarly consider a sequence $s_n \in \text{int}(E_i)$ with $s_n \rightarrow s^* \in L_i \setminus L_{i+1}$. As $s_n \cdot (p_i - p_i') \rightarrow 0$ we
must have \( \gamma_{s_n} \uparrow 1 \). Observe that the map \( \Phi : \text{int}(E_i) \to (0,1) \) given by \( \Phi(s) = \gamma_s \) is continuous. Since \( \text{int}(E_i) \) is connected, it follows that \( \Phi(\text{int}(E_i)) \) is connected. Moreover, the image contains a pair of points \( \gamma_{s_1}, \gamma_{s_2} \) with \( \gamma_{s_1} > \gamma > \gamma_{s_2} \). It follows that there exists \( s^* \in \text{int}(E_i) \) with \( \gamma_{s^*} = \gamma \). Hence, \( s^* \cdot (q - q') = 0 \). It follows that \( \text{sp}(s^*, L_i \cap L_{i+1}) \subseteq \{ s : s \cdot (q - q') = 0 \} := L_q \). Since \( s^* \in \text{int}(E_i) \) (and hence \( s^* \notin L_i \cap L_{i+1} \)), it follows that the linear dimension of \( \text{sp}(s^*, L_i \cap L_{i+1}) \) is \( k - 1 \). Since this equals the linear dimension of \( L_q \) we obtain equality. This shows that normals \( q - q' \), where \( q \) is a convex combination of \( p_i, p_{i+1} \) arise from planes of the form \( \text{sp}(s, L_i \cap L_{i+1}) \). The converse is obvious since for any \( s \in \text{int}(E_i) \) we put \( q_s := \gamma_s p_i + (1 - \gamma_s) p_{i+1} \) and obtain \( q_s - q_s' \) as a normal.

Note that the lemma gives a way to order states along a single dimension, viz. their respective MRS’s. Every hyperplane that intersects the interior of cell \( E_i \) can be associated with a unique \( \gamma \), since each such plane \( L \) is expressible as \( L := \text{sp}(s, E_i \cap E_{i+1}) \) and each \( s \in \text{int}(E_i) \) is associated with a unique \( \gamma \in [0,1] \). We first prove that \( \pi(E_1) \geq 0 \), the argument for other partition cells is similar.

**Step 3iva:** Constructing incentive compatible perturbations of the menu \( M \).

This sub-section constructs a class of incentive-compatible perturbations, which generalizes the construction in steps 3ii, 3iii. This generalization is used in a subsequent sub-step to prove positivity of the set of measures. Note that any plane \( L \) which contains the equivalence class, \([s_u]\), of the normative state must also contain the equivalence class of the “anti-normative” state, i.e. the (equivalence class of) element of \( S \) which generates the same vNM preference as \(-s_u\). Label the cell which contains this class as \( E_{-1} \). Any plane which intersects the interior of \( E_1 \) as contains the class \([s_u]\) must also intersect the interior of \( E_{-1} \), since it contains the class of \([-s_u]\). Let \( \{p_{i_1} - p'_{i_1}, \ldots, p_{i_k} - p'_{i_k}\} \) be an enumeration of the planes supporting the cell \( E_1 \), i.e. all states in \( E_1 \) pick \( p_{i_j} \) over \( p'_{i_j} \). Let \( \Sigma(p_{i_j}) = \{E_i : s \cdot (p_{i_j} - p'_{i_j}) < 0, \forall s \in E_i\} \). Note
that every adjacent cell $E_i$ is in a unique $\Sigma(p_{ij})$. Pick two planes $\{p_{ij} - p'_{ij}, p_{ik} - p'_{ik}\}$.

Group the cells adjacent to $E_1$ into three groups,

i. $\Sigma(p_{ij}) \cap \Sigma(p_{ik})^c$,

ii. $\Sigma(p_{ij})^c \cap \Sigma(p_{ik})$, and

iii. $\Sigma(p_{ij})^c \cap \Sigma(p_{ik})^c$.

Note that all cells in groups (i), (ii) lie in one of the cones, $\{s : s \cdot (p_{ik} - p'_{ik}) \geq 0, s \cdot (p_{ij} - p'_{ij}) < 0\}, \{s : s \cdot (p_{ik} - p'_{ik}) < 0, s \cdot (p_{ij} - p'_{ij}) \geq 0\}$. In either case, the intersection with the cone, $\{s : s \cdot (p_{ij} - p'_{ij}) \geq 0, s \cdot (p_{ik} - p'_{ik}) \geq 0\}$, is trivial. Let $s_u$ be the normative state ($E_1$) and consider the plane $\mathcal{L}_{su} := sp(s_u, \mathcal{L}_{pij} \cap \mathcal{L}_{pik})$.

Let $p_{su} - p'_{su}$ be a normal to this plane. Consider the cone $\mathcal{C}(\mathcal{L}_{pij}, \mathcal{L}_{su}) = \{s : s \cdot (p_{ij} - p'_{ij}) \geq 0, s \cdot (p_{su} - p'_{su}) \geq 0\}$. Find $s \in \text{int}(E_1)$ (e.g. take a small $\varepsilon$ ball around $s_u$ in $E_1$ and pick an $s$ in this ball) such that:

a. the plane $\mathcal{L} := sp(s, \mathcal{L}_{pij} \cap \mathcal{L}_{su})$ has trivial intersection with cells in groups (i), (ii).\textsuperscript{32}

b. $s \in \{\hat{s} : \hat{s} \cdot (p_{ij} - p'_{ij}) \geq 0, \hat{s} \cdot (p_{su} - p'_{su}) < 0\}$.

By the preceding lemma, find $q = \gamma \cdot (p_{su} - p'_{su}) + (1 - \gamma) \cdot (p_{ij} - p'_{ij})$ such that $\mathcal{L} = \{s : s \cdot (q - q') = 0\}$. Put $\mathcal{C}(\mathcal{L}_{pij}, \mathcal{L}) := \{s : s \cdot (p_{ij} - p'_{ij}) \geq 0, s \cdot (q - q') \geq 0\}$.

By choice of $s$, we have $\mathcal{C}(\mathcal{L}_{pij}, \mathcal{L}) \cap E_i = \emptyset$. This implies that, for any $s \in \text{int}(E_i)$, note that $s_a \in \mathcal{L}$ by linearity. Put $s_a = a_0 s + \sum_{i \neq 0} a_i s_i, s_i \in \mathcal{L}_{pij} \cap \mathcal{L}_{su}$. Note that we must have $a_0 \neq 0$ since $0 < \alpha < 1$, so that $s_a \cdot (p_{su} - p'_{su}) > 0$. On the other hand, since $s_a \cdot (p_{ij} - p'_{ij}) = 0$, $a_0 \neq 0$ implies that $s \cdot (p_{ij} - p'_{ij}) = 0$, a contradiction.

\textsuperscript{32}Reason this condition must hold for any $s \in \text{int}(E_1)$: If $\mathcal{L} \cap E_i \neq \emptyset$, then consider a putative $s'$ in this intersection (and say that $E_i$ is in group (i), the argument for group (ii) is symmetric). Find $a \in (0, 1)$ such that $\alpha \cdot s + (1 - \alpha) \cdot s' \cdot (p_{ij} - p'_{ij}) = 0$ (and, for brevity, put $s_a := \alpha \cdot s + (1 - \alpha) \cdot s'$).

Note that $s_a \in \mathcal{L}$ by linearity. Put $s_a = a_0 s + \sum_{i \neq 0} a_i s_i, s_i \in \mathcal{L}_{pij} \cap \mathcal{L}_{su}$. Note that we must have $a_0 \neq 0$ since $0 < \alpha < 1$, so that $s_a \cdot (p_{su} - p'_{su}) > 0$. On the other hand, since $s_a \cdot (p_{ij} - p'_{ij}) = 0$, $a_0 \neq 0$ implies that $s \cdot (p_{ij} - p'_{ij}) = 0$, a contradiction.
taking the plane $L' := \text{sp}(s, L_{p_{ij}} \cap L_{s_u})$, and setting $q_s - q'_s$ as the normal to this plane (where $q_s = \gamma_s \cdot (p_{su} - p'_{su}) + (1 - \gamma_s) \cdot (p_{ij} - p'_{ij})$) we must have

$$
(1 - \gamma_s) / \gamma_s > (1 - \gamma) / \gamma.
$$

Consider the following perturbation of $p_{su}$, $\hat{p}_{su} = p_{su} + \alpha_i \cdot (p_{ij} - p'_{ij})$, where $\alpha_i = (1 - \gamma) / \gamma$. For any states in cells $E_i$ within groups (i), (ii), no state $s \in \text{int}(E_i)$ which prefers $p_{su}$ to $p'_{su}$ will switch to $\hat{p}_{su}$. Also note that all states in cell $E_1$ which prefer $p_{su}$ to $p'_{su}$ now select $\hat{p}_{su}$. Now similarly perturb $p'_{su}$. Find a state in a small ball around $s_u$ such that:

1. the plane $L := \text{sp}(s, L_{p_{ik}} \cap L_{s_u})$ has trivial intersection with cells in groups (i), (ii).
2. $s \in \{ s : s \cdot (p_{ik} - p'_{ik}) \geq 0, s \cdot (p_{su} - p'_{su}) \geq 0 \}$.

By the preceding lemma, we can express the normal to $L$ as $q - q'$, where $q = \gamma \cdot (p_{su} - p'_{su}) + (1 - \gamma) \cdot (p_{ik} - p'_{ik})$. Consider the perturbed lottery $\hat{p}'_{su} := p'_{su} + \beta \cdot (p_{ik} - p'_{ik})$, where $\beta = (1 - \gamma) / \gamma$. As above, for any state $s \in E_i$, where $E_i$ is in group (ii) (since a switch to the perturbed lottery only occurs at this state if $s \cdot (p_{ik} - p'_{ik})$), we find that (taking the plane $L' = \text{sp}(s, L_{p_{ij}} \cap L_{p_{ik}})$) the rate (MRS) at which this state swaps $p_{ik} - p'_{ik}$ for $p_{su} - p'_{su}$ is $(1 - \gamma_s) / \gamma_s$, where $(1 - \gamma_s) / \gamma_s > (1 - \gamma) / \gamma = \beta$. Hence, the choice in this state remains $p_{su}$. It follows that incentive compatibility holds for all cells in groups (i),(ii).

We now need to add hyperplanes to the partition for cells in group (iii). For each $E_i$ in this group, let $L_{pi}$ denote the (unique) plane which separates cell $E_1$ from cell $E_i$ (so that $s \cdot (p_i - p'_i) < 0, \forall s \in E_i$). Consider the intersection $L_{p_{ij}} \cap L_{p_{ik}}$ and for any

---

33 Reason (for the displayed inequality): $\gamma > \gamma_s$.

34 Recall that the cone $\{ s : s \cdot (p_{su} - p'_{su}) \leq 0, s \cdot (p_{ik} - p'_{ik}) \geq 0 \}$ only contains states from cell $E_1$ or adjacent cells in group (iii).
state \( s \in \text{int}(E_1) \) consider the plane \( \mathcal{L} := \text{sp}(s, \mathcal{L}_{p_{ij}} \cap \mathcal{L}_{p_i}) \). Express the normal to this plane as \( p - p' \). Note that (i) the plane \( \mathcal{L} \) has empty intersection with \( \text{int}(E_i) \), so that the cone, \( \{ s : s \cdot (p_{ij} - p'_{ij}) \geq 0, s \cdot (p - p') \geq 0 \} \) also has empty intersection with cell \( E_i \). Now consider a state \( s' \in \text{int}(E_1) \) in a ball around \( s \) and a corresponding plane \( \mathcal{L}' := \text{sp}(s', \mathcal{L}_{p_i} \cap \mathcal{L}_{p_{ij}}) \) such that \( \{ s : s \cdot (p - p') \geq 0, s \cdot (p_{ij} - p'_{ij}) \geq 0 \} \) intersects \( E_i \) trivially (so that all states in \( E_i \) select \( p' \) over \( p \)). Writing the normal to \( \mathcal{L} \) as \( q - q' \), where \( q = \gamma \cdot p + (1 - \gamma) \cdot p_{ij} \), we have \( \gamma > \gamma_s \), for any \( s \in \text{int}(E_i) \). Consider the perturbation \( \hat{p} = p_i + \alpha_i \cdot (p_{ij} - p_i) \), where \( \alpha_i = (1 - \gamma)/\gamma \). As above, for all states \( s \in \text{int}(E_i) \) we have \( (1 - \gamma_s)/\gamma_s > (1 - \gamma)/\gamma \) (where we write \( q_s = \gamma_s \cdot p + (1 - \gamma_s) \cdot p_{ij} \) and \( q_s - q_i \) is normal to \( \mathcal{L}' = \text{sp}(s, \mathcal{L} \cap \mathcal{L}_{p_{ij}}) \)). Hence, as in the preceding argument, no states \( s \in \text{int}(E_i) \) do not switch to \( \hat{p} \). Keep track of the normal to \( \mathcal{L}' \) via \( \mathcal{L}_{q_i} \).

We see that the only states in cell \( E_i \) that make the switch to \( \hat{p} \) lie in the cone, \( \{ s : s \cdot (p_{ij} - p'_{ij}) \geq 0, s \cdot (q - q') \geq 0 \} \). We repeat this construction for every \( E_i \) in group (iii). Denote dependence on \( i \) via \( \hat{p}(i), q(i), \mathcal{L}_{q(i)} \).

Doing this for every cell \( E_i \) in group (iii) and collecting the corresponding planes and associated perturbations, we have the following list of lotteries:

a. For cells in group (i) and (ii), a single plane \( \mathcal{L}_{s_u} \) with normal \( p_{s_u} - p'_{s_u} \) and corresponding perturbed lotteries \( \hat{p}_{s_u}, \hat{p}'_{s_u} \).

b. For each cell in group (iii), a single plane \( \mathcal{L}(i) \) (we insert dependence on the cell here) with normal \( p(i) - p'(i) \) and corresponding perturbed lottery, \( \hat{p}(i) \).

c. Let \( \alpha_s = \min \{ \alpha, \{ \alpha_i \}_i \} \) denote the minimal perturbation constant tabulated across the \( \hat{p}_{s_u}, \hat{p}(i) \). Replace the perturbation constant for each \( \hat{p}_{s_u}, \hat{p}(i) \) with \( \alpha_s \).

d. Let \( \beta_s = \min \{ \beta, \{ \beta_i \}_i \} \) denote the minimal perturbation taken across \( \hat{p}'_{s_u} \). Replace all perturbation constants with \( \beta_s \).

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Starting with the partition \( \{E_i\} \) add the planes, \( \{L_s, \{L(i)\}_i\} \) and call the augmented partition \( \{E'_i\} \). Now construct the following descriptive representation of this partition. Let \( M \) be a descriptive representation of \( \{E_i\} \) without the additional planes added, and let \( \tilde{M} \) be a (to be described) descriptive representation of \( \{E'_i\} \).

Let \( f_i^0 \) denote the choice in cell \( E_i \) from \( M \). Now take any (interior) distribution on the lotteries, \( \{p_{su}/p'_{su}, \{p(i)/p'(i)\}_i\} \). Let \( \theta_i \) denote the weight on the choice from the pair \( \{p(i), p'(i)\} \) and \( \theta_0 \) the weight on the choice from \( \{p_{su}, p'_{su}\} \). Construct \( \tilde{M} \) as follows. Let \( f_j \) denote the choice in cell \( E'_j \) (where, by abuse, we label so that \( E'_j \) refines \( E_j \)) and put \( f_j := \theta_{-i} f_i^0 + \theta_0 \cdot p_{su}/p'_{su} + \theta_i \cdot p(i)/p'(i) \), where \( p_{su}/p_{su} \) (resp. \( p(i)/p'(i) \)) is abbreviation for the choice from the pair \( \{p_{su}, p'_{su}\} \) for states in cell \( E'_j \).

Let \( M = \{f_j\}_j \) and note that this gives a descriptive representation of \( \{E'_j\} \). Now consider the following perturbation, call it \( \tilde{M} \): (with elements \( \tilde{f}_j \). For cells \( E'_j, j \neq 1 \) (which refine \( E_j \)), \( \tilde{f}_j = f_j \). For cells \( E'_1 \) which refine \( E_1 \)\(^{35}\)

\[
\tilde{f}_1 = \begin{cases} 
\theta_{-i} f_i^0 + \theta_0 \cdot \hat{p}_{su} + \sum_i \theta_i \cdot \hat{p}(i), & \text{if } s \cdot (p_{su} - p'_{su}) \geq 0, s \cdot (p(i) - p'(i)) \geq 0, \forall i, \\
\theta_{-i} f_i^0 + \theta_0 \cdot \hat{p}'_{su} + \sum_i \theta_i \cdot \hat{p}(i), & \text{if } s \cdot (p_{su} - p'_{su}) < 0, s \cdot (p(i) - p'(i)) \geq 0, \forall i, \\
\theta_{-i} f_i^0 + \theta_0 p_{su}/p'_{su} + \sum_i \theta_i \cdot p(i)/p'(i), & \text{if } s \cdot (p(i) - p'(i)) < 0 \text{ for some } i.
\end{cases}
\]

For simplicity put \( \theta_i \equiv 1, \forall i \). This won’t yield a lottery for \( \tilde{f}_1 \), but we can divide by an appropriate constant to obtain a lottery after constructing an appropriate (non-lottery) \( \hat{f}_j \). As described above, not all assignments of lotteries to states are incentive compatible; however, the preceding arguments show that incentive compatibility holds outside of cells in \( E_1 \). On the cell \( E_1 \) there will be some deviations between lottery assignments and lottery choices, but we can describe for exactly which states IC does hold and, in these cases, what the IC choice is. This will be used to bound expected utility of the perturbed menu, which we turn to next.

\(^{35}\)We are abusing notation in using the same \( \tilde{f}_1 \) to designate the distinct lotteries assigned to distinct cells \( E'_1 \).
Step 3ivb: Bounding the expected utility of the perturbed menu.

Let \( \{ q(i_1) - q'(i_1), \ldots, q(i_m) - q'(i_m) \} \) be the (normals to) planes added to the partition, one for each cell \( E_i \) in group (iii). For any subset of indices \( \Sigma \subseteq \{ i_1, \ldots, i_m \} \) let \( \Theta(\Sigma) = \{ s : s \cdot (p(i_j) - p'(i_j)) < 0, \forall i_j \in \Sigma, s \cdot (p(i_j) - p'(i_j)) \geq 0, \forall i_j \notin \Sigma \} \). Now fix an \( s \in \Theta(\Sigma) \) and note that there is an associated \( k \)-tuple (where \( |\Sigma| = k \)) of MRS’s, \((1 - \gamma_{i_1})/\gamma_{i_1}, \ldots, (1 - \gamma_{i_k})/\gamma_{i_k}\), where we put \( \Sigma = \{ i_1, \ldots, i_k \} \) for notational simplicity. Let \( K = 1 + |\{ i : E_i \text{ is in group (iii)} \}| \). Recall that, to conserve notation, we used the same symbol \( f'_1 \) to denote the choice in any one of cells \( E'_1 \) which refine \( E_1 \). For the states in which \( f'_1 \) is chosen from the menu \( \hat{M} \) we have: (wlog consider \( s \) such that \( p_{s_{\alpha}} \succ s \ p'_{s_{\alpha}} \))

\[
\begin{align*}
(\diamond) \quad \tilde{f}'_1 \succeq s \ f_1 & \iff K \alpha_s \cdot (p_{ij} - p'_{ij}) \geq \sum_{l \cdot p(i_l) \prec s \cdot p'(i_l)} s \cdot (p'(i_l) - p(i_l)) \\
& \iff K \alpha_s \cdot (p_{ij} - p'_{ij}) \geq \sum_{l \cdot p(i_l) \prec s \cdot p'(i_l)} [(1 - \gamma_{i_l})/\gamma_{i_l}] \cdot s \cdot (p_{ij} - p'_{ij}) \\
& \iff K \alpha_s \geq \sum_{l \cdot p(i_l) \prec s \cdot p'(i_l)} (1 - \gamma_{i_l})/\gamma_{i_l}.
\end{align*}
\]

In the second-to-last step we have made the substitution \( s \cdot (p'(i_l) - p(i_l)) = [(1 - \gamma_{i_l})/\gamma_{i_l}] \cdot s \cdot (p_{ij} - p'_{ij}) \). The dependence on the normals \( p(i_l) - p'(i_l) \) is through the constants \( \gamma_{i_l}, (1 - \gamma_{i_l}) \). For each set \( \Sigma \) let \( \Theta_{IC}(\Sigma) \) denote the subset of states for which the choice of \( \tilde{f}'_1 \) over \( f'_1 \) is incentive compatible, i.e. the inequality in \( (\diamond) \Sigma \) holds for these states. Note that each of the normals \( p(i) - p'(i) \) are expressible as \( \gamma \cdot p_i + (1 - \gamma) \cdot p'_i \), where \( p_i - p'_i \) was the normal to the unique hyperplane separating cell \( E_i \) from \( E_1 \). Note that as \( \gamma \uparrow 1 \), the normals \( p(i) - p'(i) \) converge to \( p_i - p'_i \). Put

\[
\mathcal{E} := \{ s : \exists \Sigma, s \in \Theta_{IC}(\Sigma) \}
\]

as the full set of states (in \( E_1 \)) where the IC choice is to select \( \tilde{f}'_1 \) over \( f'_1 \). As we take \( \gamma \uparrow 1 \) for each plane \( p(i) - p'(i) \) defining the augmented partition \( \{ E'_1 \} \), the
set \( \{ s \in E_1 : s \cdot (p_{i_j} - p'_{i_j}) \geq 0, s \cdot (p(i) - p'(i)) \geq 0, \forall i \} \uparrow E_1 \) (in the sense of set containment). Fixing a set \( \Sigma \) and a state \( s \) put:

\[
MRS_\Sigma(s) := \sum_{i \in \Sigma} (1 - \gamma_i) / \gamma_i,
\]

where, for each \( i \in \Sigma \), state \( s \) lies on the plane \( \{ s : s \cdot (q_s - q'_s) = 0 \}, q_s = \gamma_i \cdot (p(i) - p'(i)) + (1 - \gamma_i) \cdot (p_{i_j} - p'_{i_j}) \). Observe that the subset

\[
\{ s : s \cdot (p_{i_j} - p'_{i_j}) \geq 0, s \cdot (p(i) - p'(i)) < 0, \text{iff } i \in \Sigma, MRS_\Sigma(s) \leq K_{\alpha_s} \}
\]

is a cell in the partition which refines \( \{ E'_i \} \) by adding the lotteries, \( \tilde{f}_j \). The reason is that if \( s, s' \) are in the same vNM class, so that there is (by lemma 1) a common \( s^* \in \partial S, \alpha_1, \alpha_2 \) such that \( \alpha_1 \cdot s^* + (1 - \alpha_1) \cdot 1/k = s, \alpha_2 \cdot s^* + (1 - \alpha_2) \cdot 1/k = s' \), then we have \( MRS_\Sigma(s) = MRS_\Sigma(s') \). Call this cell \( E^\Sigma \) and note that we have \( E = \cup_{\Sigma} E^\Sigma \), where the union is disjoint. Let \( \{ E''_i \} \) be the refinement of \( \{ E'_i \} \) which is obtained by adding the lotteries \( \tilde{f}_j \) to the menu \( \tilde{M} \).

Now we bound the difference in expected utilities, \( E_\pi \tilde{M} - E_\pi \tilde{M} \). By the preceding argument, \( \phi_{\tilde{M}}, \phi_{\tilde{M}}, \) agree on all cells in \( E''_i \) which refine \( E_j \), for any \( j \neq 1 \). On cells which refine \( E_1 \) the functions disagree only on states \( s \in E \). This set is a union of cells \( E^\Sigma \) and, moreover, on each such cell the difference \( \phi_{\tilde{M}} - \phi_{\tilde{M}} \) is constant. In particular, we have: (let \( E_\pi[\phi_{\tilde{M}} - \phi_{\tilde{M}}]|_{E^\Sigma} \) denote the difference restricted to the states in \( E^\Sigma \))

\[
(*) E_\pi[\phi_{\tilde{M}} - \phi_{\tilde{M}}]|_{E^\Sigma} = \pi(\Sigma) : \sum_{i \in \Sigma} u(p(i)) - u(p'(i)) + K_{\alpha_s}(u(p_{i_j}) - u(p'_{i_j})).
\]

If we now knew that \( \pi(\Sigma) \leq 0, \forall \Sigma \) (with strictness for some \( \Sigma \)), then we would obtain that,

\[
(**) E_\pi \phi_{\tilde{M}} - E_\pi \phi_{\tilde{M}} = \sum_{\Sigma} E_\pi[\phi_{\tilde{M}} - \phi_{\tilde{M}}]|_{E^\Sigma} < 0.
\]
By axiom 5 we must have $\tilde{M} \succeq \overline{M}$, and this yields a contradiction.

This would conclude the argument if $\pi(\mathcal{E}^\Sigma) < 0$. However, this claim is not generally true. What is true is the following weaker assertion (which follows from the same argument as Step 3iva). Recall that the elements of the collection \{\{p(i) - p'(i)\}_{i=1}^m\} are parameterized via $p(i) = \gamma_ip_i + (1-\gamma_i)p'_i$, so that we can parameterize the full set of such planes, \{\{p(i) - p'(i)\}_{i=1}^m\} by varying the mixing parameters \{\gamma_i\}_{i=1}^m.

**Lemma 20.** Assume $\pi(E_1) < 0$. There is a choice of parameters \{\{\gamma_i\}_{i=1}^m, \alpha_\ast\} such that the induced collection of IC sets, \{\mathcal{E}^\Sigma\}_\Sigma satisfies, $\sum_{\Sigma} \pi(\mathcal{E}^\Sigma) < 0$.

We now complete the bounding argument. Recall the decomposition (**) $E_\pi\phi_{\overline{M}} = E_\pi\phi_{\overline{M}} - E_\pi\phi_{\overline{M}} \in \sum_{\Sigma} E_\pi[\phi_{\overline{M}} - \phi_{\overline{M}}]|_{E_\pi} = \pi(\mathcal{E}^\Sigma), \sum_{\Sigma} u(p(i)) - u(p'(i)) + K\alpha_\ast(u(p_i) - u(p'_i))$. The terms $K\alpha_\ast(u(p_i) - u(p'_i))$ are all positive and common to every summand. Moreover, $p(i) - p'(i) = \gamma_i(p_i - p'_i) + (1-\gamma_i)(p_i - p'_i)$.

Notice that if we change the $\alpha_\ast$ and \{\gamma_i\} in a way that the boundaries of the IC sets don’t change, then the convex hulls of these common boundaries are the same. We will check that we can appropriately enlarge $\alpha_\ast$ and shrink \{\gamma_i\} such that the union $\mathcal{E} = \cup_{\Sigma}\mathcal{E}^\Sigma$ does not change. Note that this is a convex set since its interior consists of all states which strictly choose $\hat{f}_1$. To compute the boundary of $\mathcal{E}$ fix any $p(i) - p'(i)$ and find the boundary of the set $\mathcal{E}$ that lies in the cone $\{s : s \cdot (p_i - p'_i) \geq 0, s \cdot (p_i - p'_i) \geq 0\}$. Find $\gamma_s$ such that $s$ lies on the plane with normal, $\gamma_s(p'_i - p_i) + (1 - \gamma_s)(p_i - p'_i)$. If $s$ is on the boundary of $\mathcal{E}^\Sigma$, then it is indifferent between $\hat{f}_1$ and $\hat{f}_1$. This yields the following equality:

$$\gamma_s s \cdot (p_i - p'_i) + (1 - \gamma_s - K\alpha_\ast) s \cdot (p'_i - p_i) = 0.$$ 

Since $K\alpha_\ast$ will be a variable, set $\beta$ equal to this variable choice. Then, fixing the boundary state $s$ amounts to choosing $\gamma_s, \beta$ such that displayed equality holds for
the choice of this pair, i.e. thinking of $\beta$ as a chosen coordinate and the MRS($s$) as the constant. Call this $K_i(s)$ to denote that this is the constant arising from the intersection of the boundary of $\mathcal{E}$ and the cone, $\{s : s \cdot (p_i - p'_i) \geq 0, s \cdot (p_j - p'_j) \geq 0\}$ we have:

$$K_i(s) = \frac{(1 - \gamma_s) - \beta}{\gamma_s}$$

Solving for $\beta$ as a function of (the variable) $\gamma_s$ we find,

$$(*) \beta = (1 - \gamma_s) - K_i(s) \cdot \gamma_s.$$  

This equation shows that we can choose $\gamma_s$ small and compensate by enlarging $\beta$ without changing the boundary of the corresponding IC sets. Since all boundary IC states in the given cone have the same MRS, the displayed equality only depends on $i_l$, yielding: (for each $l \in \{1, 2, \ldots, m\}$)

$$(*)_l \beta = (1 - \gamma_{i_l}) - K_{i_l} \cdot \gamma_{i_l}.$$  

Note that $\beta$ is a common coordinate for each of the $l$ equalities. Moreover, choosing $\beta = 1$ we find (unless $K_i = 1$) that $\gamma_i = 0$. Thus, choosing $\beta$ close to 1 we can find $\gamma_{i_l} << 1/2, \forall l$ such that $(*)_l$ holds for all $l$. For this selection of parameters, denoted $(\beta^*, \{\gamma_{i_l}^*\})$, the IC set $\mathcal{E}$ is the same as with the original parameters, $(\alpha_s, \{\gamma_{i_l}\})$. For each $\Sigma$ the expected utility difference (again, ignoring the term $u(p_i) - u(p'_i)$ which is common to all $\mathcal{E}^\Sigma$) is expressible as:

$$(**) \sum_{i_l \in \Sigma} [u(p(i_l)) - u(p'(i_l))] = \sum_{i_l \in \Sigma} [\gamma_{i_l}^*(u(p'_i) - u(p_i))) + (1 - \gamma_{i_l}^*)(u(p_i) - u(p'_i)).$$

The direction of the inequality $E_\pi \phi_{\bar{M}} \geq E_\pi \phi_{\bar{M}}^{-1}$ (or vice-versa) is not affected by scaling the quantities $u(p_i) - u(p'_i)$. Hence, we may assume that – at the outset – we have chosen normals $p_i - p'_i$ such that $u(p_i) - u(p'_i) = 1, \forall i_l, i_j$. The term $(**)$ then reduces to,

$$(***) \sum_{i_l \in \Sigma} [(1 - \gamma_{i_l}^*) - \gamma_{i_l}^*].$$
Since we chose $\gamma^*_i << 1/2$, it follows that this quantity is positive. By making each $\gamma^*_i$ smaller if needed we can ensure: (i) positivity of $1 - 2\gamma^*_i$, (ii) $\sum_\Sigma \pi(\mathcal{E}^\Sigma)(1 - 2\gamma^*_i) < 0$, and (iii) that both $\pi(\mathcal{E})$ and $\sum_{\Sigma \neq \emptyset} \pi(\mathcal{E}^\Sigma)$ are negative. Inequality (ii) follows since (by the preceding lemma) we have $\sum_\Sigma \pi(\mathcal{E}^\Sigma) < 0$, and (iii) follows from a verbatim application of the argument in proposition 4 – since the set $\mathcal{E} \setminus (\bigcup_{\Sigma \neq \emptyset} \mathcal{E}^\Sigma)$ forms a shrinking cell in the augmented partition. Notice also that by making $\gamma^*_i$ smaller we make $\beta^*$ bigger, but this only influences the term, $\pi(\mathcal{E}^\Sigma) \cdot [\sum_{\Sigma} (u(p(i) - u(p'(i))) + K\alpha \cdot (u(p_{ij}) - u(p'_{ij})))]$, by increasing the coefficient of $u(p_{ij}) - u(p'_{ij})$. But this coefficient is common to all events $\mathcal{E}^\Sigma$. Putting everything together, we have shown that, for $\gamma^*_i$ (i.e. close to 0) sufficiently small and $\beta^*$ (i.e. close to 1) we have (factoring out the common value $u(p_{ij}) - u(p'_{ij})$)

$$E_\pi \phi_{\tilde{M}} - E_\pi \phi_{\tilde{M}} = \pi(\mathcal{E})\beta^* + \sum_{\Sigma \neq \emptyset} \pi(\mathcal{E}^\Sigma)(1 - 2\gamma^*_i) < 0.$$ 

On the other hand, by axiom 5 we must have $\tilde{M} \succeq \tilde{M}$. By the Bewley representation for (any) $\Sigma_{E_i}$-measurable menus, we must then have $E_\pi \phi_{\tilde{M}} \succeq E_\pi \phi_{\tilde{M}}$ – contradiction. This concludes the positivity argument.

**Step 4: Extension of representation to all menus**

**Step 4i: Countable Additivity**

The preceding step shows that the collection of (positive) measures $\{\Pi_{E_i}\}_{E_i}$ taken across all coherent Borel partitions patch together (canonically) to result in a single set of measures $\Pi$, each of which is measurable w.r.t. the algebra, $\mathcal{A}$.

36This mimics the argument in Proposition 4, adapted to the case where partitions are “k-wise” conical, i.e., where the intersection of any k-hyperplanes forming the partition are the same. Every Borel partition can be realized as a k-wise conical partition, e.g. take k to be the number of hyperplanes defining the partition. Consider the minimal k for which the partition is k-wise conical and apply the construction in Step 3iva in place of lemma 17.
generated by all Borel partitions. We now extend the measures comprising \( \Pi \), which are \textit{a priori} just finitely additive measures on \( A \) to countably additive (probability) measures on \( \sigma(A) \). This follows once we show two things. First, for any decreasing sequence \( \{A_n\} \), \( A_n \in A \), with \( \cap_n A_n = \emptyset \) we have \( \lim_n \mu(A_n) = 0 \) (i.e. vanishing tail probabilities). Second, for any decreasing sequence \( \{A_n\} \) (with possibly non-empty limiting intersection) let \( A = \cap_n A_n \). Any extension of \( \mu \) to \( \sigma(A) \) would have to satisfy \( \mu(A) = \lim_n \mu(A_n) \). Thus, the second thing to check is that this equality is well-defined. For this to be well-defined, the limit has to be the same no matter what sequence \( \{A_n\} \) we select. Let \( \Pi \) denote the set of a priori finitely additive (positive) measures defined on \( A \).

**Lemma 21.** Each \( \mu \in \Pi \) extends to a countably additive (probability) measure with domain \( \sigma(A) \).

\textit{Proof of Lemma 21.} We need to check (i) tail continuity and (ii) extend the measures to the sigma-closure of \( A \). Tail continuity follows from proposition 4 (convergence on shrinking cells) and positivity. Since shrinking cones have vanishing mass, for any shrinking sequence of cells we find a nesting sequence of shrinking cones and apply the proposition. Since the measures are positive, this implies the original sequence of cells must have vanishing mass as well. It follows that \( \mu(A_n) \to 0 \) whenever \( A_n \downarrow \emptyset \). Therefore, we just need to check that \( \mu \) admits a well-defined extension \( \sigma(A) \). Take any sequence \( A_n \downarrow A \) (where \( A_n \) are measurable w.r.t. some finite partition \( \{E_i(n)\} \)) and define \( \mu(A) := \lim_n \mu(A_n) \). We claim this is well-defined. Take any other sequence \( B_n \) with \( \cap_n B_n = A \) and note that we can, by taking \( A_n \cup B_n =: C_n \) if necessary, assume that \( B_n, A_n \) are measurable w.r.t. the same partition \( \{E_i(n)\} \) and that \( A_n \subseteq B_n, \forall n \). Consider the sequence of differences, \( D_n := B_n \setminus A_n \). Since \( \cap_n D_n = \emptyset \), tail continuity implies that \( \mu(D_n) \to 0 \). Hence, since \( \mu(C_n) = \mu(A_n) + \mu(D_n) \), we obtain \( \lim_n \mu(C_n) = \lim_n \mu(A_n) \). Since \( \mu(B_n) \leq \mu(C_n) \) (as \( \mu \) is positive) this implies \( \lim_n \mu(B_n) \leq \lim_n \mu(A_n) \). Reversing
the argument with the roles of $A_n, B_n$ switched gives the opposite inequality. Hence, 
\[ \lim_{n} \mu(B_n) = \lim_{n} \mu(A_n), \]
showing that $\mu(A)$ is well-defined. \qed

**Step 4ii: Representation on all menus.**

We now have a set of countably additive (probability) measures, $\Pi$, and a Bewley representation (using these measures) for all menus measurable w.r.t. the algebra $\mathcal{A}$. Now extend the representation to menus measurable w.r.t. $\sigma(\mathcal{A})$, i.e. all menus. We need to show that

\[ (*) \, M \succeq M' \iff E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi \in \Pi \]

where $\phi_M, \phi_{M'}$ are the Strotzian value functions associated to the menus $M, M'$.\(^{37}\)

Towards showing this equality (on all menus) we first prove a result characterizing convergence on the menus domain with convergence on the domain of value functions.

**Proposition 5.** Assume $M_n \to M$ and that each $M_n$ is a polytope. Then, convergence is regular if and only if $\phi_{M_n} \to \phi_M$ pointwise.\(^{38}\)

**Proof of Proposition 5.** Recall that the definition of regularity restricts to closed, (finitely) convex menus $M_n$. Show the left-to-right direction via contraposition. Assume $\phi_{M_n} \not\to \phi_M$. Find a state $s$ such that $\phi_M(s) > \lim_n \phi_{M_n}(s)$. Let $p, q \in \arg \max_{x \in M} u_s(x)$ be such that $u(p) > u(q) = \lim \sup_n \phi_{M_n}(s)$. Since $M_n \to M$, there is a $q \in M$ which attains the lim sup. Since $p \in \text{ext}(M)$ and extreme points of $M_n$ converge to extreme points of $M$, let $p_n \in \text{ext}(M_n)$ be such that $p_n \to p$. Also select $q_n \in \arg \max_{x \in M_n} u_s(x), u(q_n) \uparrow u(q)$. Since $M_n$ are polytopes, regularity

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\(^{37}\)For this argument, we don't need to transfer the preference on menus to a preference over Strotzian value functions. We will, of course, use the representation on $\mathcal{A}$-measurable menus to derive $(\ast)$ and – for this fact – we needed to transfer to ex post utility space.

\(^{38}\)The left-to-right direction does not require that $M_n$ are polytopes, but the right-to-left direction does.
implies we can select the \( p_n \to p \) such that the pair \((p_n, q_n)\) has the property that 
\[
\text{co}(p_n, q_n) \cap \text{relint}(M_n) = \emptyset \quad \text{and} \quad p_n - q_n = \lambda_n \cdot (p - q), \quad \lambda_n \neq 0 \quad \text{for all } n \text{ large}. \]
Since \( p, q \in \text{arg max}_{x \in M} u_s(x) \), this implies \( s \cdot p_n = s \cdot q_n \), contradiction.

Now consider the right-to-left direction. Fix \( p, q \in \text{ext}(M) \) with \( u(p) > u(q) \). Assume \( \phi_{M_n} \to \phi_M \) point-wise. By point-wise convergence, for each state \( s \in S \) we can find an \( N_\varepsilon \) such that \( |\phi_{M_n}(s) - \phi_M(s)| < \varepsilon, \forall n \geq N_\varepsilon \). Consider the plane \( L_{p,q} := \{ s \in S : s \cdot (p - q) = 0 \} \) and consider the compact set, \( K := \{ s \in S : d(s, L_{p,q}) \leq \delta_\varepsilon, s \cdot (p - q) \geq 0 \} \) (where \( d(\cdot, L_{p,q}) \) denotes Euclidean distance from the point to the plane \( L \)). On \( K \) there is a uniform \( N_\varepsilon \) such that: \( |\phi_{M_n}(s) - \phi_M(s)| < \varepsilon, \forall n \geq N_\varepsilon, \forall s \in K \). Towards contradiction, say that \( M_n \to M \) is not regular, so that there is a sequence \((p_n, q_n) \to (p, q)\) for which \( p_n, q_n \in \text{ext}(M_n) \) with \( \text{co}(p_n, q_n) \cap \text{relint}(M_n) = \emptyset, \forall n \text{ large}, \) and \( p_n - q_n \not\parallel p - q \forall n \text{ large} \). Consider the cone of states, \( C(L_{p_n,q_n}, L_{p,q}) = \{ s \in S : s \cdot (p_n - q_n) < 0, s \cdot (p - q) \geq 0 \} \). By hypothesis, \( C(L_{p_n,q_n}, L_{p,q}) \neq \emptyset, \forall n \text{ large} \). Moreover, by the fact that \( p_n, q_n \) are extreme points – hence, non-redundant – we know that \( p_n, q_n \) are chosen in some states (from menu \( M_n \)) and that there are states for which these two lotteries are maximal from \( M_n \) and indifferent to each other. Hence, there are states \( s \in S \) for which the maximal lottery is \( q_n \) and which border the plane \( L_{p_n,q_n} \). Note that if one of these states were in the cone \( C(L_{p_n,q_n}, L_{p,q}) \), then we would be done as \( \phi_{M_n}(s) = u(q_n) \), yet \( s \in K \) for \( n \text{ large} \) – which contradicts \( |\phi_{M_n}(s) - \phi_M(s)| < \varepsilon \). To see that there must be such a state note that if \( M_n \) were the doubleton \( \{p_n, q_n\} \) the claim would be obvious. Since \( M_n \) is finitely convex (i.e. is a polytope), the cone \( C(L_{p_n,q_n}, L_{p,q}) \) is cut by only finitely many hyperplanes, so that there must be a cell which lies within the cone in which \( q_n \) is the maximal choice from \( M_n \).

We can now extend the Bewley representation to all menus.

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\(^{39}\)For example, pick \( p_n \) to be the vertex of the polytope \( \text{con}(M_n \setminus q_n) \) which minimizes distance to the plane \( \{ x \in \Delta(X) : u_s(x) = u_s(q_n) \} \).
Proposition 6. $M \succeq M' \iff E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi \in \Pi, \forall M, M' \in \mathcal{M}$.

Proof of Proposition 6. Consider the right-to-left direction first. Assume $E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi$ and let (resp.) $\phi_{M_n}, \phi_{M'_n}$ denote Strotzian value functions attached to sequences of finite menus, $M_n$ (resp. $M'_n$) each chosen such that $\phi_{M_n} \to \phi_M, \phi_{M'_n} \to \phi_{M'}$. That is, we claim that we can choose the sequence so that $M_n$ converges to $M$ regularly. This is straightforward but involves some details, hence is set aside as a separate lemma below. Fixing an $\varepsilon > 0$, for each $\pi \in \Pi$ there is (since the $2\Pi$ are countably additive and the integrands are uniformly bounded on $S$) $N_\pi(\varepsilon)$ such that
\[ \left| E_\pi \phi_{M_n} - E_\pi \phi_M \right| < \varepsilon, \left| E_\pi \phi_{M'_n} - E_\pi \phi_{M'} \right| < \varepsilon, \forall n \geq N_\pi(\varepsilon). \]

Since the topology of weak convergence on the set $\Pi$ is metrizable (as the measures are probability measures, the test functions are bounded, and the domain is compact) we can find (in the domain $\Pi$) a $\delta_\varepsilon > 0$ such that
\[ (** \quad \left| E_\pi \phi_{M_n} - E_\pi \phi_M \right| < \varepsilon, \left| E_\pi \phi_{M'_n} - E_\pi \phi_{M'} \right| < \varepsilon, \forall \pi \text{ s.t. } d(\hat{\pi}, \pi) < \delta_\varepsilon, \forall n \geq N_\pi(\varepsilon). \]

The inequalities in (**) imply that since $\Pi$ is compact, there is a single $N_\varepsilon$ (large enough) that works for all states.\(^{40}\) Hence, for each $\varepsilon > 0$ we can find a $N_\varepsilon > 0$ such that
\[ (***) \left| E_\pi \phi_{M_n} - E_\pi \phi_M \right| < \varepsilon, \left| E_\pi \phi_{M'_n} - E_\pi \phi_{M'} \right| < \varepsilon, \forall \pi \in \Pi. \]

Now mixing each of $M, M'$ with a small weight on a $u$-maximal (resp. $u$-minimal) lottery we can assume that the inequalities $E_\pi \phi_M \geq E_\pi \phi_{M'}$ are strict for all $\pi \in \Pi$. By compactness, find $\varepsilon > 0$ such that $E_\pi \phi_M - E_\pi \phi_{M'} > 2\varepsilon, \forall \pi \in \Pi$. For this choice of $\varepsilon$ find $N_\varepsilon$ such that (***) holds. It follows that we have: (for all $n \geq N_\varepsilon$)
\[ E_\pi \phi_{M_n} \geq E_\pi \phi_M - \varepsilon > E_\pi \phi_{M'}, \forall n \geq N_\varepsilon, \forall \pi \in \Pi. \]

\(^{40}\) The metric, $d$, on $\Delta(S)$ is defined as follows. Let $\mathcal{B}(S)$ denoted the space of all (uniformly bounded) upper semi-continuous (real-valued) functions on $S$ (measurable w.r.t $\sigma(A)$). Put $d(\pi, \pi') := \sup_{f \in \mathcal{B}(S)} |E_\pi f - E_{\pi'} f|$. This metrizes the topology of weak convergence.

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Now take limits on \( n \) and use the previous proposition. Since \( \phi_{M_n} \to \phi_M, \phi_{M'_n} \to \phi_{M'} \), we have that \( M_n \to M, M'_n \to M' \) converges regularly. By the Bewley representation on \( \mathcal{A} \), we have \( M_n \succeq M'_n < \forall n \geq N_\varepsilon \). By axiom 3, we have \( M \succeq M' \). Since we have (abusing notation) replaced the original \( M, M' \) with (small) mixtures of these menus with lotteries, we then take the weights on these lotteries to zero and apply axiom 3 again to obtain \( M \succeq M' \) for the original pair of menus for which we had \( E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi \in \Pi \).

Now consider the left-to-right direction. Assume \( M \succeq M' \). Let \( M_n, M'_n \) be finite menus which converge regularly to \( M \) (resp. \( M' \)) as above and consider (using the same compactness argument), for any fixed \( \varepsilon > 0 \), a large enough \( N_\varepsilon \) such that (i) \( |E_\pi \phi_{M_n} - E_\pi \phi_M| < \varepsilon, \forall n \geq N_\varepsilon, \forall \pi \in \Pi \), (ii) \( |E_\pi \phi_{M'_n} - E_\pi \phi_{M'}| < \varepsilon, \forall n \geq N_\varepsilon, \forall \pi \in \Pi \). It follows that \( E_\pi \phi_{M_n} + \varepsilon \geq E_\pi \phi_M, \forall n \geq N_\varepsilon, \forall \pi \in \Pi \) and \( E_\pi \phi_{M'} \geq E_\pi \phi_{M'_n} - \varepsilon, \forall n \geq N_\varepsilon, \forall \pi \in \Pi \). It follows that
\[
E_\pi \phi_{M_n} + 2\varepsilon \geq E_\pi \phi_{M'_n}, \forall n \geq N_\varepsilon, \forall \pi \in \Pi.
\]

Given \( \varepsilon \), choose \( \alpha \) such that, for \( u \)-maximal \( \ell \), \((1-\alpha)u(\ell/2) = 2\varepsilon \). Then, \( E_\pi \phi_{M_n} + 2\varepsilon = E_\pi \phi_{\alpha \cdot M_n + (1-\alpha)2\ell} \). Put \( M_n^\alpha := \alpha \cdot M_n + (1-\alpha)\ell \). We have \( E_\pi \phi_{M_n^\alpha} \geq E_\pi \phi_{M'_n}, \forall \pi, \forall n \geq N_\varepsilon \). Take limits on \( n \) and obtain \( E_\pi \phi_{M^\alpha} \geq E_\pi \phi_{M'}, \forall \pi \), where \( M^\alpha := \alpha \cdot M + (1-\alpha)\ell \). Take \( \alpha \uparrow 1 \) and obtain \( E_\pi \phi_M \geq E_\pi \phi_{M'}, \forall \pi \). This concludes the proof of the proposition, and hence the proof of theorem 1.

The following lemma is used in the proof of the proposition.

**Lemma 22.** For any \( M \in \mathcal{M} \) there is a sequence \( M_n \) of polytopes such that \( M_n \to M \) regularly.

**Proof of Lemma 22.** By the preceding proposition it is sufficient to find polytopes such that the value functions \( \phi_{M_n} \) converge point-wise to \( \phi_M \). First note that, for states \( s \) such that \( u(x) = u(y), \forall x, y \in \arg \max_{z \in M} u_s(z) \), point-wise convergence
holds irrespective of the choice of sequence \( M_n \to M \). By Berge’s theorem, the correspondence \( \text{arg max}_{z \in M_n} u_z(z) \) is upper semi-continuous. Fix any \( M_n \to M \). For this fixed sequence consider any pair \((x_n, y_n)\) with \( x_n \in M_n \) and \( y_n \in \text{arg max}_{z \in M_n} u_z(z) \) we can find a convergent subsequence \((x_{n_k}, y_{n_k})\) with \( x_{n_k} \to x \in M \) and \( y_{n_k} \to y \in \text{arg max}_{z \in M} u_z(z) \). Since \( u(x) = u(y) \), \( \forall x, y \in \text{arg max}_{z \in M} u_z(z) \), every sequence of pairs \((x_n, y_n)\) has a convergent subsequence whose limiting value \( u(y) \) is the same. It follows that, choosing a sequence \((x_n, y_n)\) with \( y_n \) converging to \( \lim \inf_n \phi_{M_n}(s) \), we have that \( \lim \phi_{M_n}(s) \) exists and equals \( \phi_M(s) \). Let \( S_0 \) denote the subset of states for which the \( \text{arg max} \) on \( M \) does not lie on a single hyperplane w.r.t. the normative state \( s_u \), i.e. \( S_0 := \{ s \in S : \exists x, y \in \text{arg max}_{z \in M} u_z(z) \text{ s.t. } u(x) \neq u(y) \} \). Consider the set \( \cup_{s \in S_0} \text{arg max}_{z \in M} u_z(z) \). We now make a countable selection \( \{ x_i \}_{i=1}^{\infty} \), i.e. \( \{ x_i \}_{i=1}^{\infty} \subseteq \cup_{s \in S_0} \text{arg max}_{z \in M} u_z(z) \text{ s.t. } (*) \{ x_i \}_{i=1}^{\infty} \cap \text{arg max}_{z \in M} u_z(z) \neq \emptyset, \forall s \in S_0 \). Let \( y_i \) be any countable dense subset of \( M \) and put \( M_n = \bigcup_{i=1}^{n} x_i \bigcup_{i=1}^{n} y_i \). By the preceding argument, for any \( s \in S \setminus S_0 \), we have \( \phi_{M_n}(s) \to \phi_M(s) \). Moreover, for \( s \in S_0 \), there is some \( N_s \) large enough such that \( x_{N_s} \in \text{arg max}_{z \in M} u_z(z) \cap M_n, \forall n \geq N_s \). Hence, \( \phi_{M_n}(s) = \phi_{M_{N_s}} = u(x_{N_s}) = \phi_M(s), \forall n \geq N_s \). \( \square \)

**Proof of Theorem 2.** Let \(( S_1, B(S_1), \Pi_1 \), \(( S_2, B(S_2), \Pi_2 \) denote two subjective Bewley representations. The claim about the uniqueness of the utility kernel on lotteries is obvious, so we omit this from the argument. By hypothesis, we have \( S_i \subseteq S_{\text{NM}} \). Let \( \{ \phi_M : M \in M \} \) denote the set of all Strotzian value functions on the domain \( S_{\text{NM}} \). Let \( \phi_{M \mid S_i} \) denote the restrictions of these functions to the state spaces \( S_1, S_2 \). Let \( U \subseteq R \). The key equality is the following:

\[
(\phi_{M \mid S_i})^{-1}(U) = \phi_{M \mid S_i}^{-1}(U) \cap S_i.
\]

Let \( \sigma(\phi_{M \mid S_i}) \) denote the \( \sigma \) field (on \( S_i \)) generated by the sets \((\phi_{M \mid S_i})^{-1}(U) \). It follows that we have

\[
\sigma(\phi_{M \mid S_i}) = \{ A \subseteq S_i : A = B \cap S_i, B \in \sigma(\phi_M) \}.
\]
Hence, the (Borel) σ-field on $S_i$ is obtained from the field on $S_{vNM}$ by just intersecting each of its elements with the smaller state space, $S_i$. Let $\mathcal{B}(S_i)$ (resp. $\mathcal{B}(S_{vNM})$) denote the smallest σ-fields on $S_i$ (resp. $S_{vNM}$) which makes all the functions $\phi_M|_{S_i}$ (resp. $\phi_M$) Borel measurable and note that we have the equality,

\[(*) \quad \mathcal{B}(S_i) = \mathcal{B}(S_{vNM}) \cap S_i.\]

For any $\pi \in \Pi_1$ we must have $\mathcal{B}(S_i)$ in the domain of $\pi$. This then allows us to extend (if necessary) the domain of $\pi$ to include $\mathcal{B}(S_{vNM})$. Formally, let $\kappa : S_i \hookrightarrow S_{vNM}$ be the inclusion map and define

$$\pi^*(A) := \pi(\kappa^{-1}(A)), \forall A \in \mathcal{B}(S_{vNM}).$$

Via the above equality $(*)$ this gives a well-defined extension of the measure $\pi$ (it is easily verified that the extension $\pi^*$ is indeed a probability measure if $\pi$ is a p.m.). Hence, we now have (by replacing $S_i$ with $S_{vNM}$ and all the $\pi$’s with extended $\pi^*$’s as needed) a common state space and a common domain, viz. $S_{vNM}, \mathcal{B}(S_{vNM})$, for all measures in the respective sets $\Pi_1, \Pi_2$. Temporarily suppressing the state space and σ-fields from the description of the representation, we have two representations of $\succeq$, (i) $(u, \Pi_1)$ and (ii) $(u, \Pi_2)$. Note also that, via the map $S \rightarrow S_{vNM}$, we can pull back the measures $\Pi$ to measures on $\mathcal{B}(S)$ (as the functions $\phi_M$ are constant on vNM equivalence classes of states). Consider the algebra $\mathcal{A}$ which is the union of all finite Borel partitions $\{E_i\}$ of the state space $S$ and let $\sigma(\Sigma_{E_i})$ denote the (finite) field which is the coarsest domain of measurability for functions $\phi_M$ in $\Sigma_{E_i}$. By the uniqueness result for the “finite subjective Bewley” model (i.e. Sub-Step 3iii) we have that the set of measures $\Pi_1, \Pi_2$ are in agreement when restricted to $\sigma(\Sigma_{E_i})$. Hence, they are in agreement on all of $\mathcal{A}$. Since $\mathcal{A}$ generates $\mathcal{B}(S)$ it follows (by Dynkin’s $\pi - \lambda$ theorem) that $\Pi_1 = \Pi_2$. This shows that the extended measures are in agreement, when extended to the common domain $\mathcal{B}(S_{vNM})$. To conclude, we verify that w.r.t. these extensions the sets $S_2 \setminus S_1, S_1 \setminus S_2$ each have $\pi$-measure zero,
for all \( \pi \in \Pi \) (where \( \Pi = \Pi_1 = \Pi_2 \), is the common – by the preceding step – set of priors). But this is obvious from the way the extensions were defined. For each \( \pi_1 \) (i.e. in the original, unextended set of measures) we put \( \pi_1^* (A) = \pi_1 (\kappa^{-1} (A)) \) (where \( \kappa^{-1} (A) = A \cap S_1 \)). In particular, this means \( \pi_1^* (S_1) = 1 \) and, similarly, for any \( \pi_2 \in \Pi_2 \) we have \( \pi_2^* (S_2) = 1 \). It follows that, \( \forall \pi \in \Pi \), we have \( \pi (S_1 \setminus S_2) = 0 = \pi (S_2 \setminus S_1) \).

Note that there is a subtle issue that arises here if the sets \( S_i \) are not themselves in \( B (S_{vNM}) \), i.e. if the restricted states spaces are non-Borel subsets of the full state space of all vNM preferences. In this case, the sets \( S_1 \setminus S_2, S_2 \setminus S_1 \) may not be in the domain of the (extended) measures \( \pi \). The conclusion is then modified to all \( \pi \)-measurable subsets of \( S_2 \setminus S_1 \) and \( S_1 \setminus S_2 \).

\[ \Pi_2 \subseteq \Pi_1 \Rightarrow \text{spread}_{(u_1, S_1, \Pi_1)} (M) \supseteq \text{spread}_{(u_2, S_2, \Pi_2)} (M), \]

is obvious. Consider the left-to-right direction and assume \( \text{spread}_{(u_1, S_1, \Pi_1)} (M) \supseteq \text{spread}_{(u_2, S_2, \Pi_2)} (M) \). Taking \( M \) singleton (i.e. constant menus) we find \( u_1 = u_2 \). Equality of the state spaces (almost everywhere) will follow once we show equality of the priors, since these will be supported on the same universal space \( S \) constructed in the proof of the representation and by pulling back the measures on \( S \) to (resp.) \( S_1, S_2 \) we show that these states are equal almost everywhere. To show equality of priors, proceed via contradiction and assume that \( \Pi_2 \not\subseteq \Pi_1 \). Let \( \pi_2 \in \Pi_2 \setminus \Pi_1 \) and consider the disjoint, convex, and closed sets \( \Pi_1, \{ \pi_2 \} \). By the separating hyperplane theorem (applied to sets in \( \mathcal{L} (\mathbb{R}^n, \mathbb{R}) \)) we can find a linear functional \( \ell \in \mathcal{L} (\mathcal{L} (\mathbb{R}^n, \mathbb{R}), \mathbb{R}) \) such that

\[ \ell (\pi_1) < \ell (\pi_2), \forall \pi_1 \in \Pi_1. \]

Since \( \mathbb{R}^n \) is reflexive, the functional \( \ell \) is given by point evaluation for some \( x \in \mathbb{R}^n \), i.e. \( \ell (\pi) = \pi_1 \cdot x \), where the operation is the dot product. Hence, thinking of \( x \) as a
utility act (call it \( f_x \)) we obtain an act such that \( E_\pi f_x < E_\pi f_x, \forall \pi \in \Pi_1 \). Notice that, if \( f_x \) was a Strotzian value function (for a menu \( M_{f_x} \)), then we would be done since the inequality implies \( \text{spread}_{(u,S,\Pi_1)}(M_{f_x}) \gtrless \text{spread}_{(u,S,\Pi_2)}(M_{f_x}) \) — contradiction. This argument does not directly apply here since \( f_x \) may not be a Strotzian value function. To this end, put \( \Theta_{E_i} := \{ f \in \Sigma_{E_i} : E_\pi f > E_\pi f, \forall \pi \in \Pi_1 \} \). Also recall that \( U_{E_i} \) denotes the subset of \( \{E_i\}\)-measurable functions which are Strotzian value functions. Put

\[
\hat{U}_{E_i} = \{ f \in \Sigma_{E_i} : \exists \alpha \in [0, 1], \phi_M \in U_{E_i}, c \in \mathbb{R}_+, d \in \mathbb{R} \text{ s.t. } \alpha \cdot c \phi_M + (1 - \alpha) \cdot f = d \mathbf{I} \}.
\]

In words, \( \hat{U}_{E_i} \) consists of those \( \{E_i\}\)-measurable functions which form a perfect hedge with some (appropriately scaled) Strotzian value function.

The basic structure of the forthcoming argument is to show, towards contradiction, that if there is some \( f \in \Theta_{E_i} \), then we can form a perfect hedge (after perturbing \( f \) with a small \( \varepsilon \) if necessary) with a scaled multiple of \( f \) and some Strotzian value function. This will imply that if, say, the scalar multiple of \( f \) forming the perfect hedge is positive, that the Strotzian value function has a \( \pi \)-value strictly below its \( \pi \)-value for all \( \pi_1 \in \Pi_1 \) — contradicting the hypothesis on the comparative spreads of the menus across the two representations. The key step in this argument is the implication that \( \Theta_{E_i} \) being non-empty implies the presence of such a perfect hedge. For this we require a separation argument which uses a particular subset of \( \hat{U}_{E_i} \).

Consider the basis of indicators \( 1_{E_1}, 1_{E_2}, \ldots, 1_{E_n} \) and scale by \( \varepsilon \) such that the vectors \( \varepsilon_i := \varepsilon \cdot 1_{E_i} \) all have the property that \( f \pm \varepsilon_i \in \Theta_{E_i} \). An application of Berge’s theorem (applied to the set \( \arg \min_{v \in B_{2\varepsilon}(0)} E_\pi f_n \pm v \)) also shows that as \( f_n \rightarrow f \) (and halving \( \varepsilon \) to allow a \( 2\varepsilon \) ball, if needed) we have \( f_n \pm v \in \Theta_{E_i}, \forall n \geq N, \forall v \in B_{2\varepsilon}(0) \). Find \( K_1 >> 0 \) such that for all \( K \geq K_1 \) we have \( (1 - 1/K) \cdot f + v \in \Theta_{E_i} \) for all \( v \in B_{2\varepsilon}(0) \). Take \( \varepsilon'_i := \frac{1}{K_1} \cdot \varepsilon_i, \forall i \) and note that we have \( \varepsilon'_i \in B_{2\varepsilon}(0) \).

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Take any \( f \in \Theta_E \), for which (i) \( \max_i |f_i| \leq 1 \) and (ii) \( \sum_i f_i = 0 \) (n.b. such an \( f \) always exists whenever \( \Theta_E \neq \emptyset \)). Let \( \Sigma^0_{E_i} \subseteq \Sigma_{E_i} \) be the subset of non-constant Strotzian value functions. Put \( K := \{ f + \varepsilon_i \}_{i} \) and consider the following set

\[
\mathcal{V} := \cup_{c \geq 1} c \cdot \Sigma^0_{E_i} + \cup_{d \in \mathbb{Z}} d \cdot K.
\]

Put \( \mathcal{D} := \{ d \cdot \mathbf{1} : d \in [-1, 1] \} \).

**Lemma 23.** If \( \Theta_E \neq \emptyset \), then \( \text{co}(\mathcal{V}) \cap \mathcal{D} \neq \emptyset \).

n.b. From this implication, we will derive a contradiction, which then implies that \( \Theta_E \) must be empty.

**Proof of Lemma 23.** Towards contradiction, if the intersection were empty we could find some \( p \neq 0 \) such that

\[
p \cdot v \leq p \cdot d, \forall v \in \text{co}(\mathcal{V}), \forall d \in \mathcal{D}.
\]

Since \( \mathbf{0} \in \mathcal{D} \) this implies that the LHS is upper-bounded by 0. On the other hand, note that – by construction – the vectors \( f + \varepsilon_i \) comprising \( K \) are linearly independent. Hence, \( p \cdot (f + \varepsilon_i) < 0 \) for at least one such \( i \). Pick any \( \phi_M \in \Sigma_{E_i} \) and choose \( d << 0 \) such that \( p \cdot [d \cdot (f + \varepsilon_i)] + p \cdot \phi_M > 0 \). By construction, \( d \cdot (f + \varepsilon_i) \) for \( \phi_M \in \mathcal{V} \). On the other hand, the inequality contradicts separation. Hence, we must have \( \text{co}(\mathcal{V}) \cap \mathcal{D} \neq \emptyset \). \( \Box \)

Let \( d^* \mathbf{1} \) be a putative element in the intersection and \( c_1 \phi_{M_1} + d_1 k_1, \ldots, c_l \phi_{M_l} + d_l k_l \in \mathcal{V} \) be such that \( \sum_{i=1}^l \alpha_i \cdot (c_i \phi_{M_i} + d_i k_i) = d^* \mathbf{1} \), for some \( \alpha_i \in [0,1] \) with \( \sum_i \alpha_i = 1 \). Let \( d_i^+ \) denote those \( i \) for which \( d_i \) is positive and similarly denote \( d_i^− \). Also define \( d^+_a := \sum \alpha_i d_i^+ = \sum \alpha_i d_i^- = d_a = \sum \alpha_i d_i \). Notice that we have \( \sum \alpha_i c_i \phi_{M_i} = (\sum \alpha_i c_i) \cdot (\sum \hat{c}_i \phi_{M_i}) \), where we put \( \hat{c}_i := c_i / (\sum \alpha_i c_i) \). Since \( \Sigma^0_{E_i} \) is convex, it follows that \( \sum \hat{c}_i \phi_{M_i} \in \Sigma_{E_i} \). Hence, \( \sum \alpha_i c_i \phi_{M_i} = (\sum \alpha_i c_i) \cdot (\sum \hat{c}_i \phi_{M_i}) \in \cup_{c > 0} c \cdot \Sigma_{E_i} \). It follows that \( \sum_i \alpha_i (c_i \phi_{M_i} + d_i) = c \phi_M + \sum_i \alpha_i d_i \), for \( \phi_M \in \Sigma_{E_i} \). We now consider the latter sum. Consider five cases,
i. \( d_a^+ > |d_a^-|, d_a^+ / |d_a^-| \leq K_1 \),

ii. \( d_a^+ > |d_a^-|, d_a^+ / |d_a^-| > K_1 \),

iii. \( d_a^- < |d_a^-|, d_a^- / |d_a^-| \leq K_1 \),

iv. \( d_a^- < |d_a^-|, d_a^- / |d_a^-| > K_1 \),

v. \( d_a^+ = |d_a^-| \).

Cases (iii)-(iv) are entirely symmetric to cases (i)-(ii), hence we just provide the argument for the former. Case (v) requires a separate argument.

**Case (i)**: \( d_a^+ \geq 2|d_a^-|, d_a^+ / |d_a^-| \leq K_1. \)

Let \( k_i = f + \varepsilon_i \) and note that

\[
\sum_i \alpha_i d_i (f + \varepsilon_i) = |d_a^-| \left[ \frac{d_a^-}{|d_a^-|} \cdot f + \frac{d_a^+}{|d_a^+|} \cdot \sum_{i: d_i > 0} \alpha_i d_i \varepsilon_i \right] + \sum_{i: d_i < 0} \alpha_i d_i \varepsilon_i, \]

Notice that \( \frac{d_a^-}{|d_a^-|} \cdot f \in \Theta_E \). Moreover, \( \frac{d_a^-}{|d_a^-|} = \frac{d_a^+}{d_a^+} - 1 \geq 1 \) so that the term in brackets is also in \( \Theta_E \). Hence, denoting this term as \( g \), we have that \( c\phi_M + |d_a^-| g \) is a constant act. Since \( g \in \Theta_E \), this implies that \( \phi_M \in -\Theta_E \), i.e. \( E_\pi \phi_M < E_{\pi_1} \phi_M, \forall \pi_1 \in \Pi_1 \).

This contradicts the hypothesis that \( \text{spread}_{(u,\Pi_2)}(M) \subseteq \text{spread}_{(u,\Pi_1)}(M) \).

**Case (ii)**: \( d_a^+ \geq 2|d_a^-|, d_a^+ / |d_a^-| > K_1 \).

In this case, factor out \( d_a^+ \) and note that we have

\[
\sum_i \alpha_i d_i (f + \varepsilon_i) = d_a^+ \left[ \frac{d_a^-}{|d_a^-|} \cdot f + \frac{d_a^+}{|d_a^+|} \cdot \sum_{i: d_i > 0} \alpha_i d_i \varepsilon_i \right] + \frac{1}{|d_a^-|} \sum_{i: d_i < 0} \alpha_i d_i \varepsilon_i. \]

\[\text{We are assuming, additionally, that } d_a^+ \geq 2|d_a^-|. \text{ Relaxing this hypothesis requires an argument similar to the one for case (v), hence we address this possibility when we address case (v).}\]
Note that \( \frac{d\alpha}{d\alpha} = 1 - \frac{d\alpha}{d\alpha} \leq 1 - \frac{1}{K_1} \), so that \( \frac{d\alpha}{d\alpha} f \in \Theta_{E_i} \). Moreover, all such elements (by choice of \( K_1 \)) have the property that \( (1 - \frac{1}{K_1}) f + v \in \Theta_{E_i}, \forall v \in B_{2\varepsilon}(0) \). Since

\[
\frac{d\alpha^+}{|d\alpha^+|} \sum_{i,d_i>0} \frac{\alpha_i d_i}{|d\alpha^+|} \varepsilon_i' + \frac{1}{|d\alpha^+|} \sum_{i,d_i<0} \alpha_i d_i \varepsilon_i' \in B_{2\varepsilon}(0)
\]

it follows that \( \sum_i \alpha_i d_i (f + \varepsilon_i) \) is a scalar multiple of an element in \( \Theta_{E_i} \), so that the same argument in case (i) applies.

For cases (iii)-(iv), i.e where we take \( |d\alpha^-| > d\alpha^+ \), \( |d\alpha^-| \geq K_1 \cdot d\alpha^+ \) (resp. \( |d\alpha^-| < K_1 \cdot d\alpha^+ \)) proceed as follows. Note that for any \( f \in \Theta_{E_i} \) we have \( f \in \Theta_{E_i} \). Find \( K_1 \) large enough (and an \( \hat{\varepsilon} \)) such that \( -(1 - \frac{1}{K_1}) f \in -\Theta_{E_i} \) for any \( K \geq K_1 \) and, moreover, \( -(1 - \frac{1}{K_1}) f + v \in -\Theta_{E_i}, \forall v \in B_{2\varepsilon}(0) \). Now choose \( K_1 \) to be the maximum of \( K_1, \hat{K}_1 \) and choose the minimum of \( \varepsilon, \hat{\varepsilon} \). Using \( K_1, \varepsilon \) we apply the argument for cases (i)-(ii) verbatim to cases (iii)-(iv) (n.b. \( d\alpha \) is negative in cases (iii)-(iv)). The remaining cases are:

a. \( d\alpha = 0 \),
b. \( d\alpha \neq 0 \) and \( d\alpha^+ > d\alpha^- \) and \( d\alpha^+ < 2d\alpha^- \),
c. \( d\alpha^- > d\alpha^+ \), \( d\alpha^- < 2d\alpha^+ \).

For these cases, pass to a particular subset of \( \Sigma_{E_i}^0 \). Consider the set of \( \Sigma_{E_i}^0 \)-measurable \( \phi_M \) such that: (the labelled constants \( K_1, K_2 \) are unrelated to the previous arguments)

1. \( \phi_M(s) \geq K_1, \forall s \in \cup_{j=1}^{i} E_j \).
2. \( \phi_M(s) \leq -K_2, \forall s \in \cup_{j=i+1}^{n} E_j \).
3. \( \phi_M(s_1) \geq 2|\phi_M(s_2)|, \forall (s_1, s_2) \in (\cup_{j=1}^{i} E_j) \times (\cup_{j=i+1}^{n} E_j) \).

where we take \( K_1 >> 2K_2 >> 1 \). Let \( \Sigma_{E_i}^{1} \subseteq \Sigma_{E_i}^{0} \) denote the set of such \( \phi_M \). Notice that \( \Sigma_{E_i}^{1} \) is obviously closed and convex. It is also non-empty. To see this, let
\{L_1, \ldots, L_k\} enumerate the hyperplanes defining the partition, where \(L_i := \{s : s \cdot (p_i - p'_i) = 0\}\), and label so that \(\bigcup_{j=1}^i E_j = \{s : s \cdot (p_i - p'_i) \geq 0\}\). Take \(M\) to be a descriptive representation of \(\{E_i\}\) and perturb \(p_i, p'_i\) by (resp.) \(p_i + \gamma_1 \cdot (p_i - p'_i), p'_i + \gamma_2 \cdot (p'_i - p_i), \gamma_1, \gamma_2 > 0\), i.e. for each \(f \in M\) we replace with \(\hat{f}\) where the slot in which \(p_i\) (resp. \(p'_i\)) occurs in the lottery defining \(f\) is replaced with \(p_i + \gamma_1 \cdot (p_i - p'_i)\) (resp. \(p'_i + \gamma_2 \cdot (p'_i - p_i)\)). The resulting menu may not be composed of lotteries, but this will not have relevance for the argument. If we can find a perfect hedge between \(f \in \Theta_{E_i}\) and some \(\phi_M\), where \(M\) has perturbed lotteries, then we mix with \(\frac{1}{k}\) to obtain a perfect hedge between \(\alpha \cdot f\) and the mixed menu – which is a standard menu of lotteries.

Now consider, as in cases (i)-(iv), the set \(V := \bigcup_{c \geq 1} c \cdot \Sigma_{E_i} + \bigcup_{d \in \mathbb{Z}} d \cdot K\). We claim that \(\text{co}(V) \cap D \neq \emptyset\). Else, the verbatim separation argument as in cases (i)-(iv) (viz. lemma 19) yields a contradiction. Consider a putative element in the intersection and note that, since \(\bigcup_{c \geq 1} c \cdot \Sigma_{E_i}\) is convex, we reduce to considering elements \(\sum_i \alpha_i d_i (f + \varepsilon\alpha) = d \alpha f + \sum_i \alpha_i d_i \varepsilon\alpha\). Consider (sub)case (a).

**Lemma 24.** \(d\alpha \neq 0\).

**Proof of Lemma 24.** Wlog we take \(\varepsilon\alpha := \varepsilon \cdot 1_{E_i}\), i.e. the indicators \(\varepsilon\), differ only by a common scale. We will be taking sums across all components \(i\) in the argument below and this common \(\varepsilon\) will factor out of the sum, hence we will suppress it in what follows. If \(d\alpha = 0\), then for \(j \geq i+1\) we must have \(\alpha_j d_j + c \cdot \phi_M(s) = d \in [-1, 1]\). Hence,

\[
\alpha_j d_j \leq -c \cdot \phi_M(s) + 1, \forall s \in E_j.
\]

Similarly, for any \(j \leq i\) we have \(\alpha_j d_i + c \cdot \phi_M(s) = d\), so that \(\alpha_j d_j \leq -c \cdot \phi_M(s) + 1\) (recall that \(\phi_M(s) \geq K_1\), so that \(d_j < 0\)). Since \(\phi_M(s_1) \geq 2|\phi_M(s_2)|, \forall (s_1, s_2) \in (\bigcup_{j \leq i} E_j) \times (\bigcup_{j \geq i+1} E_j)\), we have

\[
\alpha_j d_j \leq -c \cdot \phi_M(s_1) + 1 \leq 2c \cdot \phi_M(s_2) + 1, \forall s_2 \in \bigcup_{j \geq i+1} E_j.
\]
Notice that we can select a grouping \{E_1, \ldots, E_i\}, \{E_{i+1}, \ldots, E_n\} (at the outset) such that \(i \geq n - i\). This can be done by choosing a hyperplane in the collection \(\{L_1, \ldots, L_k\}\) such that at least half the cells comprising the partition lie on the side of the hyperplane \(L_i\) given by \(\{s : s \cdot (p_i - p_i') \geq 0\}\). If, for each hyperplane \(L_i\) in the collection, more than half the cells in the partition \(\{E_i\}\) lie on the side \(\{s : (p_i - p_i') \leq 0\}\), then we consider \(\Sigma_{E_i}^2\) given by the set of all \(\phi_M\) such that \(\phi_M(s_2) \leq 2|\phi_M(s_1)|, \forall (s_1, s_2) \in (\cup_{j \leq i} E_j) \times (\cup_{j \geq i+1} E_j)\). The symmetric argument as the one given below applies to this set. Hence, we assume (wlog) that \(\{E_1, \ldots, E_i\}\) has larger cardinality than \(\{E_{i+1}, \ldots, E_n\}\). For each \(j \geq i + 1\) pick any \(s_2' \in E_j\) and note that we have, \(\forall s \in E_i\) \((i \leq j), \phi_M(s) \leq 2\phi_M(s_2') + 1\). Hence, for each \(j \geq i + 1\) assign an \(j_i \leq j\) (with associated state \(s_i^j \in E_i\)), and note that for the (possibly) remaining \(j \leq i\) we have \(\phi_M(s) \leq 2\phi_M(s_2')\) (where we put \(s_2' \in \cup_{j \geq i+1} E_j\) a state realizing to the max of \(\{\phi_M(s_2'^{i+1}), \ldots, \phi_M(s_2^*)\}\)). Adding over all \(j\) we have

\[
0 = d_n = \sum_{j \leq i} \alpha_j d_j + \sum_{j > i+1} \alpha_j d_j
\]

\[
\leq \sum_{j=i+1}^n [2\phi_M(s_2^j) - \phi_M(s_i^j)] + 2 + \max\{i - (n - i), 0\}[2\phi_M(s_2^*) + 1]
\]

\[
< 0.
\]

The latter inequality since \(K_2 \gg 1\), so that \((n - i)\phi_M(s_2^*) < -2(n - i)\). This shows that we must have \(d_n \neq 0\). 

Now consider the possibility that \(\text{co}(\mathcal{V}) \cap \mathcal{D} \neq \emptyset\) and that an element in the putative intersection is such that either \(d_\alpha < d_\alpha^+ < 2d_\alpha\) or \(d_\alpha^- < d_\alpha^- < 2d_\alpha^+\). Both cases rely on the following reduction. First introduce some simplifying notation. Consider the “positive” states for the functions in \(\Sigma_{E_i}^1\), i.e. \(\{s : \phi_M(s) > 0, \forall \phi_M \in \Sigma_{E_i}^1\} = \cup_{j \leq i} E_j\) and let \(\mathcal{I}_1\) denote the indices of the cells comprising these states, viz. \(\mathcal{I}_1 = \{1, 2, \ldots, i\}\). Similarly, let \(\mathcal{I}_2\) denote the set of indices of the cells comprising the negative states. Now define \(d_{\mathcal{I}_1}^+ := \sum_{i \in \mathcal{I}_1, d_i > 0} \alpha_i d_i, d_{\mathcal{I}_1}^- := \sum_{i \in \mathcal{I}_1, d_i < 0} \alpha_i d_i\). Similarly
define $d_{I_2}^+, d_{I_2}^-$. Notice that we must have

$$ (*) \quad | \sum_{i \in I_1} [d_\alpha f_i + |d_{I_1}^+ + d_{I_1}^-|] \varepsilon | \geq 2 \sum_{i \in I_2} [d_\alpha f_i + (d_{I_2}^+ + d_{I_2}^-)] \varepsilon . $$

**Lemma 25.** $|d_{I_1}^+ + d_{I_1}^-| \geq 2(d_{I_2}^+ + d_{I_2}^-)$.

**Proof of Lemma 25.** Notice that if $\sum_{i \in I_1} d_\alpha f_i \geq 0$, this is obvious from definition of the set $\Sigma^1_{E_i}$. If $\sum_{i \in I_1} d_\alpha f_i < 0$ and (towards contradiction) we also have $|d_{I_1}^+ + d_{I_1}^-| < 2(d_{I_2}^+ + d_{I_2}^-)$, then $(*)$ implies we must have $|\sum_{i \in I_1} d_\alpha f_i| > \sum_{i \in I_2} d_\alpha f_i$. This contradicts the hypothesis that $\sum_i f_i = 0$ (which implies $|\sum_{i \in I_1} d_\alpha f_i| = \sum_{i \in I_2} d_\alpha f_i$).

We assume henceforth that

$$ -(d_{I_1}^+ + d_{I_1}^-) = |d_{I_1}^+ + d_{I_1}^-| \geq 2(d_{I_2}^+ + d_{I_2}^-). $$

**Lemma 26.** $d_\alpha^- < d_\alpha^+ < 2d_\alpha^{-42}$ cannot occur.

**Proof of Lemma 26.** Let $\phi_M + d_\alpha f_i + d_i \alpha_i \varepsilon$ denote the $i$-th coordinate of the vector in the putative intersection $co(V) \cap D$, where we put $\phi_M$ equal to the $i$th coordinate of the (scaled multiple of) Strotzian value function $\phi_M$. Since $-1 \leq \phi_M + d_\alpha f_i + d_i \alpha_i \varepsilon \leq 1$ we sum across coordinates to get: $-|\{E_i\}| \leq \sum_i \phi_M^i + d_\alpha \sum_i f_i + d_\alpha |\{E_i\}| \varepsilon \leq |\{E_i\}|$. Since $\sum_i f_i = 0$, this simplifies to:

$$ -|\{E_i\}| \leq \sum_i \phi_M^i + d_\alpha |\{E_i\}| \varepsilon \leq |\{E_i\}|. $$

Since $\phi_M \in \Sigma^1_{E_i}$ we have $\sum_i \phi_M^i > > 1$, implying that $d_\alpha < 0$. \qed

We are left to consider the case that $d_\alpha^+ < |d_\alpha^-| < 2d_\alpha^-$. Notice that in defining the subset $\Sigma^1_{E_i}$ we could have adjusted property (3) to: $\phi_M(s_1) \geq K \cdot |\phi_M(s_2)|, \forall (s_1, s_2) \in (\cup_{j \leq i} E_j) \times (\cup_{j \geq i+1} E_j), K \geq 2$ and the same argument as above applies verbatim. Hence, we now take $K > > 2$ in the definition of property (3) and consider the

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42Even $d_\alpha^- < d_\alpha^+$.  

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modified subset $\Sigma_{E_i}$. Notice that if the presumptive element in the intersection $\text{co}(\mathcal{V}) \cap \mathcal{D}$ satisfies $d^+_{i_1} < d^-_i < 2d_i$ then we have:

$$\text{(**) } |d^-_{I_1} - d^+_{I_1}| \geq K \cdot (d^+_{I_2} + d^-_{I_2}).$$

**Lemma 27.** The case $d^+_{i_1} < |d^-_i| < 2d_i$ cannot occur.

**Proof of Lemma 27.** We use (***) to obtain a pair of contradictory bounds on $d_i$. For the first bound, sum the coordinates of the vector in the intersection of $\text{co}(\mathcal{V}) \cap \mathcal{D}$ to get: (for brevity, put $k := |\{E_i\}|$)

$$\text{(\#) } \frac{\sum_i \phi^i_M - k}{\varepsilon} \leq |d_i|. \quad (43)$$

Now just sum across coordinates in the positive (resp. negative) states. This yields the following two strings of inequalities:

1. $\sum_{i \in I_1} \phi^i_M + d_i \sum_{i \in I_1} f_i - k \leq (|d^-_{I_1} - d^+_{I_1}|)\varepsilon \leq \sum_{i \in I_1} \phi^i_M + d_i \sum_{i \in I_1} f_i + k.$

2. $\sum_{i \in I_2} \phi^i_M + d_i \sum_{i \in I_2} f_i - k \leq (d^+_{I_2} + d^-_{I_2})\varepsilon \leq \sum_{i \in I_2} \phi^i_M + d_i \sum_{i \in I_2} f_i + k.$

Notice that we can wlog omit any $i$'s from the summation for which $f_i = 0$, hence assume for simplicity that $f_i \neq 0$, $\forall i$. We claim that the hypothesis that $2d_i > |d^-_i| > d^+_{i_1}$ implies that $f_i < 0, \forall i \in I_1, f_i > 0, \forall i \in I_2$. To see this, note that $\phi_i >> 0, \forall i \in I_1$ and $\phi_i << 0, \forall i \in I_2$. Since $d_i > \max\{d^+_{i_1}, |d^-_i|\}/2$ and $\varepsilon$ is a function of the given $f$ (which can be made arbitrarily small, with its upper threshold dependent on $K$, hence implicitly $f$), we can select $\varepsilon << \min_i |f_i|/2$. Hence, for each coordinate $i \in I_1$ we have $|d_i f_i| > |d_i \alpha_i|\varepsilon$. If $f_i > 0$ for any $i \in I_1$ this implies $d_i f_i + d_i \alpha_i < 0$, which then makes it impossible to have $(\phi + d_i f_i + \sum_j d_j \alpha_j \varepsilon_i)_i \in [-1, 1]$. Similarly obtain a contradiction if $f_i < 0$ for any $i \in I_2$. Hence, put $f^1_* := \max\{\max_{i \in I_1} - f_i\}, f^2_* := \max_{i \in I_2} f_i, f_* = \min\{f^1_*, f^2_*\}$. From (**), (1), (2) we have the bound,

$$K \cdot (\sum_{i \in I_2} \phi^i_M + d_i \sum_{i \in I_2} f_i - k) \leq \sum_{i \in I_1} \phi^i_M + d_i \sum_{i \in I_1} f_i + k.$$

\[\text{Given the, alleged, hypothesis that } d^+_{i_1} < |d^-_i| < 2d_i, \text{ the absolute values are superfluous.}\]
This implies the following upper bound on \(|d_\alpha|\) (from (\(\ast\ast\))):

\[
K|d_\alpha|f^* \leq \sum_{i \in I_1} \phi^i_M - K \sum_{i \in I_2} \phi^i_M + (K + 1)k
\]

Equivalently,

\[
|d_\alpha| \leq \dfrac{\sum_{i \in I_1} \phi^i_M - K \sum_{i \in I_2} \phi^i_M + (K + 1)k}{Kf^*}.
\]

Put together with (\(\spadesuit\)) we get:

\[
(\spadesuit) \dfrac{\sum_{i \in I_1} \phi^i_M - k}{\varepsilon} \leq \dfrac{\sum_{i \in I_1} \phi^i_M - K \sum_{i \in I_2} \phi^i_M + (K + 1)k}{Kf^*}.
\]

Since \(f, K\) are determined prior to selection of \(\varepsilon\) we can (for fixed choice of \(f, K\)) restrict if necessary to the sub-class of functions \(\Sigma^2_{E_i}\) in \(\Sigma^1_{E_i}\) such that, say,

\[
K^2(\sum_{i \in I_1} \phi^i_M - k) > \dfrac{\sum_{i \in I_1} \phi^i_M - K \sum_{i \in I_2} \phi^i_M + (K + 1)k}{Kf^*}.
\]

Notice that \(\Sigma^2_{E_i}\) is a convex (and non-empty) subset so that the same separation arguments apply. Now choosing any \(\varepsilon < \frac{1}{K^2}\) we get a contradiction to (\(\spadesuit\)). \(\square\)

Returning to the proof of the theorem, we have shown that if \(\Theta_{E_i} \neq \emptyset\), then \(\text{co}(V) \cap D\) must be (on the one-hand) non-empty. On the other hand, evaluating an alleged element of the intersection on a case-by-case basis, we have shown that the only possible such elements (i.e. those in cases (i)-(iv)) that could exist imply a contradiction on the premise that \(\text{spread}^{(u, \Pi_1)}(M) \subseteq \text{spread}^{(u, \Pi_2)}(M)\). Hence, \(\Theta_{E_i} = \emptyset\), implying that \(\Pi_2 \subseteq \Pi_1\). \(\square\)
References


