Abstract: This paper introduces and axiomatizes lexicographic multiple prior preferences. Our model generalizes lexicographic subjective expected utility (LSEU) (see, e.g. Blume et al. (1991a)) by allowing for uncertainty (i.e. multiple priors) in each step of the ordered chain of beliefs. This extends the applicability of lexicographic choice procedures to settings with payoff uncertainty. We give two such applications. The first is a signaling model, where the uncertainty is about the payoff types of players. The second is a problem of insurance under uncertainty, where the uncertainty is about endowments.

1 Introduction

This paper introduces and provides behavioral foundations for Lexicographic Multiple Priors Subjective Expected Utility theory (hereafter, “LMP-SEU”). This generalizes lexicographic subjective expected utility (“LSEU”), after Fishburn (1971), Fishburn (1974), and Blume et al. (1991a)), to allow for uncertainty in the chain of beliefs in the lexicographic representation. To motivate our generalization, let us briefly recall the decision-procedure represented by the LSEU model. This theory replaces a single subjective expected utility (“SEU”) model with a ladder of SEU models. The intuition is that the model on the highest rung of this ladder is the most plausible representation of the decision-maker’s (“DM’s”) choices, in the sense that it explains nearly all choices. However, there are some choices which cannot be explained by the model occupying the top rung. For cases in which this model offers no prediction on choice, we turn to the model on the next highest rung. This model resolves some of the indeterminacies in choice, but some might still remain. If there are choices which cannot be explained by either this model or the model on the top-rung, then we move to the model on the third step of the ladder. This process continues until we reach a hierarchy of SEU models which together explain all choices.

The advantage of lexicographic choice procedures is that a collection of models can, in principle, allow for a better description of choice behavior than a single model alone. However, a drawback of restricting these procedures to subjective expected utility is that it requires each agent to precisely evaluate the likelihood of all payoff-relevant events. In other words, the LSEU model does not allow agents

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to account for the presence of payoff uncertainty. This is a substantive omission. For example, in games of incomplete information players are modeled using payoff types. Player types usually influence not only their payoffs, but also those of the other players in the game. Hence, if one player is uncertain about another player’s type this should influence both her conjecture about that player’s strategy as well as her own choice of strategy. The goal of the present paper is to provide an extension of the LSEU model which allows for the presence of such uncertainty. In our paper, the word “uncertainty” (also referred to as “ambiguity” or “multiple priors”) is used in a decision-theoretic sense and denotes any choice setting which has the following two features: (i) choices map randomly to outcomes and (ii) the random process which maps choices to outcomes has an unknown distribution. When applied to games, payoff uncertainty - in this sense - denotes ambiguity about the payoff types of other players.

The LSEU model has some room for uncertainty already built-in to the model since, when used as a utility criterion in games, players are allowed to hold multiple conjectures about their opponents strategies. However, according to the LSEU representation these conjectures (beliefs) come with a strict priority-ordering. The belief on the top rung in the representation is viewed as “infinitely more likely” (see pg.84, Blume et al. (1991b)) than the belief on the next rung, and so on. Therefore, while there are multiple beliefs none of these are placed on an equal footing and, in this sense, there is no ambiguity about beliefs. The DM still makes choices as if she has a clear-eyed view of the likelihood of all events.

Ambiguity arises naturally in economic environments and to address its presence we require a different utility criterion than LSEU. To begin with, we need to take a stand on how the DM assesses ambiguity. In other words, what unique number does she assign to a lottery when lottery outcomes have several possible distributions? The criterion we apply throughout this paper is known as the max-min expected utility (MEU) criterion, after Gilboa and Schmeidler (1989). This means that a DM facing uncertainty proceeds in two steps. First, she associates a set of subjective expected utility outcomes to each choice and, second, she evaluates any given choice in terms of the worst outcome from the utility possibility set. When we assume DMs apply the MEU criterion to make decisions, the induced set of equilibria are often different than those under SEU. Moreover, the MEU criterion does not offer any guidance on selecting any one of these outcomes over the other. This is where the lexicographic multiple priors criterion (LMP-SEU) enters the story. We show through a set of examples that the LMP-SEU criterion accomplishes two objectives, i.e. it accounts for the presence of uncertainty and also delivers a unique equilibrium prediction.
Example 1.
To illustrate, we first present a standard signaling game below. When there is no uncertainty about the type of the sender, the game has a unique weak perfect Bayesian equilibrium (wPBE) outcome, which is also obtained in pure strategies. However, once we introduce a little uncertainty, there are multiple other, and less plausible, equilibria that come into existence. These equilibria all share the common feature that, to justify on-path behavior by the senders, the receiver plays a particular mixed strategy off the equilibrium path. Applying the LMP-SEU criterion, it turns out that these “off-path mixing” equilibria can all be ruled out while maintaining the original equilibrium.

Figure 1: Pre-emption game

The game bears similarity to the well-known beer-quiche game (Cho and Kreps (1987), pg.183). The players are two firms, where the sender is a potential entrant and the receiver is an incumbent. The sender has two actions, invest ($i$) or don’t invest ($di$), and can be one of two types, i.e. strong or weak. Investing is more costly for the weak firm than for the strong.\(^1\) The incumbent sees the investment decision and then decides to either accommodate (not fight, $n$) entry or to fight entry ($f$).\(^2\) The unique pure-strategy wPBE in this game is for both sender types to invest and for the incumbent to fight entry regardless of the investment decision.\(^3\)

\(^1\)This assumption is embodied in an increasing differences condition in the payoffs, i.e. fixing a receiver action, the payoff difference from investing is weakly higher for the strong type than it is for the weak type.

\(^2\)We explain the notation in the extensive-form representation. Player 2 is the receiver and player 1 is the sender. Nature, player “c”, makes the initial move and selects player 1’s type as either “strong” or “weak”. Player 1 observes nature’s move, but player 2 does not. Player 1 then makes a choice to play either $i$ or $di$. Player 2, observing this choice, then makes a subsequent choice to either play $f$ or $n$. The pair $(x, y)$ is shorthand for (sender’s payoff, receiver’s payoff).

\(^3\)There is an outcome equivalent wPBE in which the receiver mixes only off-path, but this can be ruled out by a standard refinement such as the Cho-Kreps criterion. Both of these equilibria carry over with uncertainty. However, there is also a new equilibrium in which the receiver mixes on and off-path, i.e. not only strategies but also equilibrium outcomes are non-unique under uncertainty.
Now add uncertainty to the prior probability, $p$, that the entrant is weak by letting $p \in [0.50, 0.50 + \varepsilon]$. This will directly impact the play of the receiver since, anticipating a sender strategy, she will assign a set of beliefs – as opposed to a single belief, or a collection of priority-ordered beliefs (as in LSEU) – to the type of sender who chooses that strategy. When these beliefs are evaluated using the MEU criterion, there are now multiple wPBEs: the original wPBE plus others which share the common feature that the receiver fights with some probability on path and mixes off path. Existing refinements from signaling games cannot, in general, be used to dispose of such equilibria.\(^4\)

**Example 2.**
Payoff uncertainty also arises in contexts other than games. As a second application of our model, we consider an example of a market for ex ante contracts over a contingent commodity. Agents are endowed with a single unit of a consumption commodity. There are two points in time, time 0 and time 1. Consumption takes place only at time 1, but between time 0 and time 1 there is a random shock which affects each agent's endowment of the time 1 consumption good, i.e. there is endowment risk. Agents are risk-averse, so that the presence of this shock induces a time 0 market for insurance against endowment risk. An equilibrium in this framework consists of prices for insurance contracts and the time 1 net allocations induced by the contracts purchased at time 0. Uncertainty in this context is with respect to the distribution of the random shock which affects contingent endowments. When the distribution of the shock is known to all traders, i.e. there is no uncertainty, the economy exhibits a unique Arrow-Debreu (AD) equilibrium: each agent is insured against endowment risk and the amount of insurance she gets is proportional to the probability that she receives an endowment shock. That is, the higher the risk, the lower the market price of her endowment and, consequently, the less insurance she is able to purchase.

When the endowment shock is subject to uncertainty and agents evaluate contingent contracts using the MEU criterion, this equilibrium is still maintained. However, there is now also an additional equilibrium in which the agent who, in a sense, faces the greatest endowment risk receives no gains from trade. That is, her contingent endowment is priced at 0 and she is allocated no time 1 consumption. This happens

\(^4\)For example, for the given cardinal payoffs, the Cho-Kreps criterion (Cho and Kreps (1987)) can be applied to rule out some, but not all, of the off-path mixing equilibria. In particular, there are equilibria with distinct on-path behavior which survive this refinement. Moreover, if we change the payoff for the strong sender at the (don't invest, fight) node to, say, 4 rather than 3.6, we maintain the same equilibria. However, for this modified game the Cho-Kreps criterion does not eliminate any of the equilibria. On the other hand, applying the LMP-SEU criterion restores uniqueness in both cases.
despite the fact that the other agents demand a positive amount of the excluded agent’s endowment and would each be strictly worse off if they could not do so. For this reason, we call this a “free insurance” equilibrium.\(^5\) In terms of a market prediction, equilibria in which some goods are free seem less plausible than those in which prices are positive. As in the signaling game, we show that replacing the MEU criterion with the LMP-SEU criterion selects the positive prices outcome and restores uniqueness.

There are some differences between the examples both in the arguments and in the definitions of equilibria. In the next section, we develop both examples in detail and show that, by applying the LMP-SEU criterion, we recover the equilibrium under no uncertainty while eliminating the others that arise under the MEU criterion.\(^6\) After discussing these examples, we present some axioms and turn to our main result, which is a representation theorem for the LMP-SEU model. While our axioms will look familiar to readers who have seen the Blume et al. (1991a) and Gilboa and Schmeidler (1989) papers, there are some new ideas involved in deriving the representation from these axioms. In section 3, we state our representation theorem and give a sketch of some of the details which are novel to our model. Section 4 concludes and the appendix collects proofs omitted from the main text.

2 Axioms, Examples, and Model

The formal definition of the model is laid out (in section 2.2) in a general decision-theoretic setting. We turn to this only after we show how the model can be used to refine outcomes of signaling games and trading behavior in markets. To understand these examples, it is not necessary to read the formal definition first. We have already described the LSEU model. Relative to this model, the LMP-SEU model makes two changes. First, in each step of the lexicographic chain it replaces a single prior with a set of priors. Second, at each level of the chain it evaluates a given strategy or consumption bundle by assigning the lowest expected utility from amongst the priors at that level, i.e. we apply the MEU criterion.

\(^5\)That this outcome is an equilibrium requires, in particular, that it must be individually rational for the agent allocated no consumption to participate in this market. The reason this outcome is still individually rational for this agent is that, under the MEU criterion, she also anticipates receiving zero utility from her endowment. This pessimism is evidently what is being reflected in the equilibrium price for her endowment, even though other agents do obtain value from trading with this agent.

\(^6\)Since the MEU model is nested in the LMP-SEU model, our definition of an LMP-SEU equilibrium (for both examples) will use a refinement of the LMP-SEU representation. The refinement (see definitions 2, 4) imposes three restrictions on the general definition of an LMP-SEU model. First, the chain of beliefs in the LMP-SEU representation has length greater than 1. Second, beliefs are getting less uncertain along the chain (i.e. lower-priority beliefs are subsets of higher-priority beliefs). Third, best-response (resp. demand) correspondences are not constant along the chain.
2.1 Examples

Set-up for example 1

We list the wPE in the pre-emption game. We show in the appendix that there are no separating equilibria. In total, there are three kinds of wPBE, all of which involve senders pooling: (proofs omitted here are in the appendix)

- **Equilibrium 1.** Both senders invest and the receiver fights on and off-path (i.e. invest, fight entry).
- **Equilibrium 2.** Both senders invest and the receiver fights on path and mixes off-path.
- **Equilibrium 3.** Both senders invest and the receiver mixes on and off-path (i.e. invest, mix on and off-path).

Note that the multiplicity of equilibrium is not just in off path play, but also in (on path) outcomes themselves. Equilibria 1 and 2 are outcome equivalent. However, in equilibrium 3 the receiver’s equilibrium behavior is not the same in as in the others. That is to say, when we introduce uncertainty and players use the MEU criterion to evaluate payoffs, both equilibrium outcomes and strategies are indeterminate.

The forthcoming definitions and arguments invoke some notation particular to the signaling game. We summarize for the reader’s benefit.

- Receiver (pure) actions: \( n_f = \text{“not fight”} \) (relabelled here to avoid confusion with the index \( n \)), \( f = \text{“fight”} \).
- Sender (pure) actions: \( i = \text{“invest”}, \ di = \text{“don’t invest”} \).
- \( \Pi_{di}^n \) = \( n \)th-stage (in the LMP-SEU chain) of receiver’s beliefs about the sender’s type after receiving the message \( di \).
- \( \Pi_i^n = n \)th-stage of receiver’s beliefs about the sender’s type after receiving the message \( di \).

We now verify that with the LMP-SEU criterion equilibria 2 and 3 are eliminated, so that uniqueness is recovered. This first requires a definition of what it means to be an LMP-SEU equilibrium. For the receiver, a strategy \( \sigma_R \) is a triple \((a_i, a_{di}, \{\Pi_i^n, \Pi_{di}^n\}_{n=1}^N)\) consisting of a ladder of contingent beliefs, \( \{\Pi_i^n, \Pi_{di}^n\}_{n=1}^M \), and a pair of contingent action choices, where \( a_i \) is a lottery on \{\( f, n_f \)\} and the subscript denotes invest/don’t invest. The sender strategy is indexed by type, \( \sigma_w, \sigma_s \), and is a lottery on \{\( di, i \)\}. Put together, a strategy profile is given by a triple \((\sigma_w, \sigma_s, \sigma_R)\). The solution concept described below is wPBE applied to our game with type uncertainty.
Definition 1. A profile \((\sigma_w, \sigma_s, \sigma_R)\) is an LMP-SEU equilibrium if the following conditions hold:

i. Each sender type’s strategy is a best response to \(\sigma_R\).

ii. On-path beliefs are obtained from the initial set of priors by prior-by-prior updating, where updates are computing using Bayes’ rule (when possible) and the anticipated sender strategy pair \((\sigma_s, \sigma_w)\).

iii. The receiver’s strategy \(\sigma_R\) is an (LMP-SEU)-best-response at each of her information sets, given sender strategies \((\sigma_w, \sigma_s)\) and her posterior beliefs.

The second condition says that beliefs are correct on the equilibrium path. Or rather, it is the analogue of this property when there is uncertainty about the sender’s type. Note that, as with wPBE, posterior beliefs are unrestricted at unreached information sets. The second and third conditions together also ensure that LMP-SEU equilibrium can be a refinement of MEU equilibrium. Without further restrictions on the belief chains it is, however, a vacuous refinement. Since the belief chains can have length 1, all MEU equilibria are nested in the set of all LMP-SEU equilibria. To make a proper refinement of this solution concept requires that the ladder of beliefs comprising the receiver strategy \(\sigma_R\) is actually relevant. Otherwise the receiver’s utility collapses to MEU (at each information set) and we cannot refine away any of the undesirable equilibria. Given the sender strategies, \((\sigma_w, \sigma_s)\) and the chain of MEU representations given by the belief sets \(\Pi_n := \{\pi^n_i, \pi^n_{di}\}\), let \(BR_n(i), BR_n(di)\) (resp. \(BR_{n+1}(i), BR_{n+1}(di)\) for short) denote the receiver’s best-response at each of her information sets. Let \(N\) be the length of the lexicographic chain.

Definition 2 (Non-redundant LMP-SEU profiles). A profile of strategies \((\sigma_w, \sigma_s, \sigma_R)\) is a non-redundant LMP-SEU profile if it satisfies the following three properties:

i. \(N \geq 2\).

ii. \((BR_n(i), BR_n(di)) \neq (BR_{n+1}(i), BR_{n+1}(di)), \forall n \leq N\).

iii. \(\Pi^n_i \supseteq \Pi^{n+1}_i, \Pi^n_{di} \supseteq \Pi^{n+1}_{di}\).

Let us explain these restrictions. Since the LMP-SEU model nests the MEU model and equilibrium is not determinate under MEU, to restore uniqueness we need to formally allow more than one level of beliefs in our LMP-SEU representations. This is what condition (i) requires. However, if under these alternative belief sets the best-responses of the receiver are identical, then there isn’t any sense in which the beliefs at different levels of the hierarchy can be relevant for equilibrium play. Condition (ii) therefore imposes the additional condition that, as we move along the chain, the best-response must be changing at some information set. Finally,
condition (iii) says that beliefs about the sender become more refined as we move along the hierarchy. Define a non-redundant LMP-SEU equilibrium to be an LMP-SEU equilibrium in which, additionally, the profile \((\sigma_w, \sigma_s, \sigma_R)\) is non-redundant.

**Example 1** (Applying LMP-SEU restores uniqueness in the pre-emption game). We first verify that the outcome where both senders invest and the receiver fights (with probability 1) on and off the equilibrium path is a non-redundant LMP-SEU equilibrium outcome. Second, we verify that there are no other non-redundant equilibria.

**Step 1**: Both senders investing and the receiver fighting on and off-path is a non-redundant LMP-SEU equilibrium.

For the first claim, put \(\sigma_w = i = \sigma_s\) and note that this determines all conditional beliefs \(\Pi_n\) along a putative equilibrium hierarchy. Consider the following receiver strategy, \(\Pi_{d1} = \{(\pi_w, \pi_s) : \pi_w \in [0.5, 0.5 + \varepsilon]\}, \Pi_{d2} = \{(\pi_w, \pi_s) : \pi_w \in [0.5 + \varepsilon/2, 0.5 + \varepsilon]\}, \sigma_R(i) = f, \sigma_R(di) = f.\) In this case, it is a strict best-response to play \(\sigma_R^d = f.\) Moreover, the profile is non-redundant, so that the “invest, always fight entry” wPBE in the MEU game is maintained under a non-redundant LMP-SEU equilibrium.

**Step 2**: There are no other non-redundant LMP-SEU equilibria.

Next we verify that there are no other non-redundant LMP-SEU equilibria. Since LMP-SEU equilibria are also MEU equilibria, we just need to check that none of the other pooling equilibria which were present under the MEU criterion can be sustained in a proper LMP-SEU equilibrium. To eliminate classes (2) and (3) it suffices to check that mixing off-path is never a weak best-response in a non-redundant LMP-SEU equilibrium. Fix a putative non-redundant profile \((\sigma_w, \sigma_s, \sigma_R)\) which sustains either (2) or (3) in equilibrium. Let \(\{\Pi_n\}\) denote the hierarchy of beliefs and let \(\rho\) denote the probability that the receiver chooses to not fight entry, and \(\pi\) the probability that the sender is the weak type. Fixing \(\rho\), as a function of the set of posteriors \(\Pi_n\) the receiver’s payoff from mixing \((\rho, 1 - \rho)\) on accommodate(not fight)/fight gives payoff:

\[*\) \(\min_{\pi \in \Pi_n} (\pi + \rho - 2\pi \rho).\)

When \(\rho \neq 1/2\), the minimum occurs (strictly) at the extreme points of the set \(\Pi_n\). When \(\rho = 1/2\), the value is independent of \(\pi\) and equals 1/2. This implies that, for any mixing on or off-path to be an equilibrium, the only viable \(\rho\) is 1/2. The equilibrium hypothesis implies that \(\Pi_n = \Pi_{n+1}, \forall n.\) Hence, given the sender strategies we must have \(BR_n(i) = BR_{n+1}(i), \forall n.\) On the other hand, by property (2) we have \((BR_n(i), BR_n(di)) \neq (BR_{n+1}(i), BR_{n+1}(di)).\) Hence, \(BR_n(di) \neq \)

\(\)Hence, this is an LMP-SEU representation with a length 2 hierarchy of beliefs.
To be able to compute equilibrium we assume the utility indices $(1, 8)$.

If the former holds, then (since $(1, 8)$ giving the receiver the same payoff independent of the posterior set) the fact that $\Pi_1^a \supseteq \Pi_2^a$ implies that $\min_{\pi_1: \pi_a \in [0, \rho]} \pi_a = 1/2$ and that $\min_{\pi_1: \pi_a \in [0, \rho]} \pi_a > 1/2$. Hence, $BR_2(di) = \{(0, 1)\}$. Condition $(2)$ defining a non-redundant LMP-SEU profile, then implies the lexicographic chain has length $N = 2$. It follows that fighting with probability $1$ is the receiver’s unique off-path best-response to the senders pooling on invest. Given this, if the receiver is playing $(1/2, 1/2)$ upon observing invest, the strong type will deviate. Hence, she is fighting with probability $1$ on and off-path and each type of sender is investing with probability $1$. The same argument implies that, if condition (ii) holds, then the chain has length $2$ and $BR_2(di) = \{(1, 0)\}$. But then, the weak type sender will deviate from investing. Hence, the only non-redundant LMP-SEU equilibrium is where both sender types invest and the receiver fights on and off-path.

Note that the length $N \geq 2$ hypothesis defining non-redundance is critical here. Without it, all profiles where the receiver holds just a single belief set (on and off-path) are proper. Hence, we need the receiver to hold a second-order conjecture about the sender’s type and for this conjecture to be different than her first-order conjecture to restore uniqueness of equilibrium. We now present a second application of the LMP-SEU criterion which shares some features of the pre-emption game, although in the context of trading in decentralized markets.

**Set-up for example 2**

Consider an exchange economy with three traders, say $a, b, c$. There is a single contingent commodity for sale at time $0$. All traders have state-independent (i.e. at time $1$) consumption preferences represented by concave utility indices $u_a(\cdot), u_b(\cdot), u_c(\cdot)$. There are three possible states of the world, say $\{s_1, s_2, s_3\}$, and state-contingent endowments are given as follows ($(x, y, z)$ denotes $x$ in state $s_1$, $y$ in state $s_2$, $z$ in state $s_3$): $\omega_a = (1, 0, 0), \omega_b = (0, 1, 0), \omega_c = (0, 0, 1)$. Trade takes place once and for all at time $0$. Objects being traded are contracts on contingent commodities and the time $0$ market is complete. Traders $a, b$ and $c$ are uncertain about how the state will resolve and have a common set of prior beliefs, $\Pi_a = \Pi_b = \Pi_c := \{\pi : \pi_1 = \pi_3 \in [0.5 - \varepsilon/2, 0.5], \pi_2 \in [0, \varepsilon]\}$. To be able to compute equilibrium we assume the utility indices $u(\cdot)$ are (i) twice differentiable and (ii) satisfy the Inada conditions, i.e.

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*Here we put $\pi = (\pi_1, \pi_2, \pi_3)$, where $\pi_1 = \Pr(s = s_1)$, and so on.*
\lim_{x \to 0} u'(x) = -\infty, \lim_{x \to \infty} u'(x) = 0. Traders facing uncertainty evaluate contingent commodities using the MEU criterion, i.e. the utility of a contingent bundle is its minimal SEU value from the set of plausible priors. In this trading environment, we compute the set of Arrow-Debreu (AD) equilibria. To fix ideas, we recall the definition of an AD equilibrium in our framework.

**Definition 3.** Let \( \succeq_a, \succeq_b, \succeq_c \) denote agents’ respective preferences over contingent commodity plans \( x = (x_{s1}, x_{s2}, x_{s3}) \). Also let \( \omega_a, \omega_b, \omega_c \) denote the agents’ contingent endowments. A pair \((p, x)\) (where \( x := (x_a, x_b, x_c) \) and \( p = (p_1, p_2, p_3) \)) is an Arrow-Debreu equilibrium if the following conditions hold:

1. (Preference maximization) For each agent, given prices \( p \) the contingent consumption bundle \( x \cdot \) is \( \succeq \)-maximal on the budget set, \( B(p, \omega) = \{ x \in \mathbb{R}_+^3 : p \cdot x \leq p \cdot \omega \} \).

2. (Market clearing) Put \( \bar{\omega} := \sum_{j} \omega_j \). Then, \( x_a + x_b + x_c = \bar{\omega} \).

The price vector \( p \) denotes prices of the three Arrow securities and demand bundles \( x(p) \) are expressed in terms of date 1 consumption.

**Step 1:** Computing AD equilibria in the no-uncertainty economy.

The no-uncertainty benchmark is where both traders share the common prior of \( \pi = (.5 - \epsilon/2, \epsilon, .5 - \epsilon/2) \). Under no uncertainty there is a unique, full-insurance AD equilibrium which we now describe. Denote commodity prices \( p = (p_1, p_2, p_3) \), where \( p_i > 0 \) and \( \sum_i p_i = 1 \). Note that we have a common prior, concave utility kernels, and no aggregate uncertainty. Hence, all Pareto optimal allocations are full insurance allocations, i.e. constant across states. Moreover, the first-welfare theorem (which applies to our context since preferences in this paper are locally non-satiated) implies that AD equilibria are Pareto optimal. Hence, the only candidate equilibria are allocations of the form, \( x_a = (p_1, p_1, p_1), x_b = (p_2, p_2, p_2), x_c = (p_3, p_3, p_3) \). It remains to check that these all solve the agents’ respective utility-maximization problems (UMP’s) at prices \( (p_1, p_2, p_3) \). To show this first note that, (i) by strong monotonicity of preferences, all AD equilibria have strictly positive prices and (ii) by the Inada conditions, for any such strictly positive price vector, each agent’s demand bundle(s) is interior. Concavity (and differentiability) of the utility kernel implies that state-dependent demand satisfies the following first-order condition (FOC) for the associated lagrangean:

\[
\frac{u'_a(x^+_a)}{u'_a(x^-_a)} = \frac{p_i/\pi_i}{p_j/\pi_j}
\]

The same FOC holds for all agents \( a, b, c \) and for all pairs of states \((i, j)\).
Now use the two facts that any candidate equilibrium is (i) Pareto-optimal (and hence a full-insurance profile), and (ii) solves the FOC (by interiority). Plugging in (ii), note that the LHS of the FOC is equal to 1 at any smooth allocation, hence we immediately obtain \( p_1 = p_3 \). Moreover, we have

\[ p_3 = \frac{\pi_3}{\pi_2} p_2. \]

Since \( \sum_i p_i = 1 \) this gives,

\[ p_2 = \frac{1}{1 + \frac{\pi_3}{\pi_2} + \frac{\pi_1}{\pi_2}}. \]

Simplifying, we obtain: \( p_2 = \pi_2 \). Hence, with no uncertainty there is a unique AD equilibrium which involves full-insurance and state-contingent prices equal the respective ex ante probabilities with which those states occur.

**Step 2:** Computing Arrow-Debreu equilibria in the economy with uncertainty.

When we introduce uncertainty, we still maintain the original AD equilibrium, but there is now an additional equilibrium in which agent \( b \) consumes nothing and her endowment is purchased (for free) by the other agents as a form of insurance. We list both equilibria below. For brevity of notation, we put \( \pi_1 = .5 - \varepsilon/2, \pi_3 = .5 + \varepsilon/2, \pi_2 = \varepsilon \).

- **Equilibrium 1.** Allocations: \( x_a = (\pi_1, \pi_1, \pi_1), x_b = (\pi_2, \pi_2, \pi_2), x_c = (\pi_3, \pi_3, \pi_3) \); Prices: \( p := (p_1, p_2, p_3) = (\pi_1, \pi_2, \pi_3) \).

- **Equilibrium 2.** Allocations: \( x_a = (1/2, 1/2, 1/2), x_c = (0, 0, 0) \); Prices: \( p = (1/2, 0, 1/2) \).

The proof of this characterization of AD equilibria is in the appendix. We refer to the second equilibrium as a “free insurance” outcome. Consider the two state economy obtained by deleting agent 2 and state \( s_2 \) from the model. Assume there is still uncertainty in the sense that the set of prior distributions over states is given by, \( \{\pi = (\pi_1, \pi_3) : \pi_1 \in [.5 - \varepsilon, .5 + \varepsilon]\} \). It turns out that the unique AD equilibrium in this two-agent economy is where agents \( a, c \) “share ambiguity” and each consumes \((1/2, 1/2)\). This is referred to as ambiguity sharing since (i) it involves a full-insurance outcome and (ii) the outcome is the same as in the no-uncertainty economy where each agent holds the common prior of \( \pi = (1/2, 1/2) \) which splits the difference in the uncertainty commonly perceived by both players. When we add agent \( b \) into the economy and endow her with the sole consumption good in state \( s_2 \), then the sole source of the uncertainty is the likelihood with which state \( s_2 \) will occur. Since, ceteris paribus, agents \( a, c \) would buy insurance to hedge against state \( s_2 \), this should bid up the value of the contingent good in state \( s_2 \). In particular,
equilibrium prices should attach positive value to agent b’s endowment.

Yet, this is not what happens in equilibrium 2. In this equilibrium, agents a and c are still fully insured against, but this insurance is purchased for free since agent b’s endowment is priced at 0. Since the LMP-SEU criterion nests the MEU criterion, both equilibria persist when we allow traders to have LMP-SEU preferences. However, when we refine this criterion in a manner similar to the pre-emption game example, equilibrium 2 is eliminated and equilibrium uniqueness is restored.

We now verify that a refinement of the LMP-SEU criterion eliminates the free insurance outcome while preserving the trade outcomes in which agents are fully insured. As in the pre-emption game example, this first requires a definition of equilibrium in which all traders have LMP-SEU preferences. The concept of equilibrium is still Arrow-Debreu, i.e. taking prices as given, an equilibrium is where agents optimize, submit demands, and markets clear. However, the primitive of an Arrow-Debreu model consists of (i) a definition of preferences over commodities and (ii) endowments. We want to use LMP-SEU to refine AD equilibria (in which agents have MEU preferences). For this reason, we need to define (as we did for the signaling game example) what it means for traders to have non-redundant LMP-SEU preferences. For the reader’s benefit we first make a list of the notation used in the definition:

- \( \{\Pi_a^i\}, \{\Pi_b^i\}, \{\Pi_c^i\} \) denotes the chain of beliefs. Put \( \Pi_a := \{\Pi_a^i\}, \Pi_b := \{\Pi_b^i\}, \Pi_c := \{\Pi_c^i\} \).
- \( D_a(p) \) (resp. \( b, c \)) denotes the given agent’s demand correspondence when her preferences are LMP-SEU and represented by the chain \( \{\Pi_a^i\} \) (resp. \( b, c \)).
- \( D_a^1(p) \) (resp. \( b, c \)) denotes the given agent’s demand when her preferences are MEU with beliefs given by \( \Pi_a^1 \) (resp. \( b, c \)).

For brevity, we have suppressed mention of the utility kernels in describing agent preferences.

**Definition 4 (Proper LMP-SEU economy).** A pair, \( \{(\Pi_a, \Pi_b, \Pi_c); (\omega_a, \omega_b, \omega_c)\} \), consisting of beliefs and endowments constitutes a proper LMP-SEU economy if the following conditions hold:

- i. Letting \( N_a, N_b, N_c \) denote the respective lengths of the lexicographic orders for the three traders, we have \( N_a, N_b, N_c \geq 2 \).
- ii. For some agent, say agent a, and some price \( p \) we must have, \( D_a(p) \neq D_a^1(p) \).
- iii. For each level \( i \) and each agent, we have \( \Pi_a^i \supseteq \Pi_a^{i+1} \) (resp. for agents \( b, c \)).
We have presented the restrictions in the same order as definition 2 (the definition of a non-redundant LMP-SEU strategy profile) to try to bring out the similarities. The idea is to find sufficient conditions on the primitives of the AD economy that guarantee that it is observably distinguishable from the companion AD economy in which we replace everyone’s chain of beliefs with just their first-stage belief. The first condition says that the chain of beliefs is a chain with more than one step, else the agent in question (vacuously) has MEU preferences. This by itself is not sufficient to guarantee that the economy is observably distinguishable from an MEU economy. The second condition says that, for some agent, the demand correspondence induced by her chain of beliefs is distinct from the demand correspondence induced by just the first stage beliefs.

This just means that the correspondences are both well-defined and disagree at some price $p$. There is no restriction on the price $p$ that allows us to distinguish these two demand correspondences. In particular, it need not be an equilibrium price. Since we only observe equilibrium prices, conditions (1) and (2) are not by themselves sufficient to ensure that the LMP-SEU economy is observably distinct from an MEU economy. This then brings us to condition (3), which says that, for each agent, uncertainty is decreasing as we move along her chain of beliefs. This condition mirrors the one in definition 2. As in that example, stage 1 beliefs represent objective ambiguity, i.e. the universe of plausible priors, and subsequent belief selections must therefore be subsets of these. These three conditions turn out to be sufficient to ensure that the LMP-SEU economy is observably distinguishable from the MEU economy (induced by the first-stage beliefs).

Example 2 (Proper LMP-SEU restores uniqueness in the AD model). We compare two Arrow-Debreu economies. In both economies contingent endowments are given by $\omega_a = (1, 0, 0), \omega_b = (0, 1, 0), \omega_c = (0, 0, 1)$ and each agent has a concave, differentiable utility kernel, $u$, which satisfies the Inada conditions. Moreover, in both economies agents have LMP-SEU preferences over contingent consumption bundles. The only difference is that in one economy (economy 1, say) each agent holds a chain of beliefs $\{\Pi^a_i\}$ (resp. for agents $b, c$) about how the state will resolve and, in the other economy (economy 2), each agent’s beliefs are just the first-stage beliefs from economy 1, i.e. $\Pi^a_i$ (resp. for $b, c$).

Our goal is to use the concept of a proper LMP-SEU economy to refine the equilibria in the MEU economy described in the set-up. That is, economy 2 is the MEU economy and economy 1 is a proper LMP-SEU in which the first-stage beliefs agree with the beliefs of agents in the MEU economy. We now show two things. First, we show that the full-insurance, no uncertainty outcome is an equilibrium in a proper

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9Our usage of the term “proper” is unrelated to the way in which it is used in Blume et al. (1991b).
LMP-SEU economy. Second, we show that there are no proper economies in which the “free insurance” outcome is an equilibrium.

**Step 1:** Showing that the no-uncertainty outcome is maintained in equilibrium in a proper LMP-SEU economy.

To show this we need to produce LMP-SEU preferences for each agent (whose first stage beliefs agree with the beliefs in economy 2, the MEU economy) which admits the no-uncertainty outcome in an AD equilibrium. Consider the following chain of beliefs (which we take to be common to all agents),

- \( \Pi_1 := \{ \pi = (\pi_1, \pi_2, \pi_3) : \pi_1 = \pi_3 \in [0.5 - \varepsilon/2, 0.5] \} \),
- \( \Pi_2 := \{ \pi = (\pi_1, \pi_2, \pi_3) : \pi_1 = \pi_3 \in [0.5 - \varepsilon/2, 0.5 - \varepsilon/4] \} \).

Note that the associated LMP-SEU economy is proper. Clearly, conditions (1) and (3) of the definition of a proper economy are satisfied. To check condition (2) we first observe that second-stage beliefs can only be relevant to break ties between first-stage optimal consumption baskets when prices are non-interior, i.e. some price is zero. To prove this, consider any positive price \( p' \). We claim that each agent’s demand correspondence is (i) well-defined and (ii) singleton. Well-defined-ness is clear since prices are positive and wealth is finite. For the latter, consider any given agent’s, say agent \( a \)’s, utility maximization problem and let \( x_a(p)(= (x^1_a(p), x^2_a(p), x^3_a(p))) \in D_a(p) \). We claim that

\[
(*) \quad u_a(x^2_a(p)) \leq 0.5 \cdot u_a(x^1_a(p)) + 0.5 \cdot u_a(x^3_a(p)).
\]

Towards contradiction, if (*) doesn’t hold then the MEU-value of the bundle \( x_a(p) \) strictly obtains on the prior \( \pi_* = (0.5, 0, 5) \). Since \( p_2 > 0 \), agent \( a \) can obtain the same utility from the basket \( (x^1_a, x^2_a, x^3_a) \), where \( x^1_a = x^1_a(p) \), \( x^3_a = x^3_a(p) \) and \( x^2_a \) is chosen so that \( u(x^2_a) = 0.5 \cdot u_a(x^1_a) + 0.5 \cdot u_a(x^3_a) \). Note that the MEU-value of the bundle \( (x^1_a, x^2_a, x^3_a) \) is the same on all first-stage measures \( \pi \in \Pi_1 \) and equals the MEU-value of \( (x^1_a(p), x^2_a(p), x^3_a(p)) \). However, agent \( a \) is not exhausting her budget at \( (x^1_a, x^2_a, x^3_a) \) (since \( p_2 > 0 \)), contradicting the hypothesis that \( x_a(p) \in D_a(p) \) (since she could sell \( x^2_a - x^2_a \) of good 2 and use the sales to purchase equal, small amounts of goods 1, 3 and, thereby, increase ex ante utility.

It follows that the solution to agent \( a \)’s UMP occurs at bundles where the MEU-value is attained on the prior \( \pi = (0.5 - \varepsilon/2, \varepsilon, 0.5 - \varepsilon/2) \). In other words, we have:

\[
D_a(p) = \arg\max_{x_a \in B(p, \omega_a)} [(0.5 - \varepsilon/2) \cdot u_a(x^1_a) + \varepsilon \cdot u_a(x^2_a) + (0.5 - \varepsilon/2) \cdot u_a(x^3_a)].
\]

The objective is a strictly concave function (on \( \mathbb{R}^3_a \)) and the constraint set is compact and convex, implying that \( D_a(p) \) is a singleton (and non-empty). Hence, to check (2)
we need only to consider boundary prices \( p \). If \( p_1 = 0 \) or \( p_3 = 0 \), \( D_a(p) \) is undefined, hence consider \( p_2 = 0 \). We check that \( D_b(p) \neq D_b^1(p) \). Note that

\[
D_b^1(p) = \{ x_b = (x_b^1, x_b^2, x_b^3) : x_b^1 = 0 = x_b^3, x_b^2 \in [0, 1] \}.
\]

Moreover, given our second stage beliefs \( \Pi_2 \) we find that \( D_b(p) = \{(0, 1, 0)\} \), so that \( D_b(p) \neq D_b^1(p) \). It follows that the LMP-SEU economy is proper. Since only first-stage beliefs matter for positive prices and the no-uncertainty equilibrium is maintained under MEU preferences, we need only check that the given LMP-SEU economy has no other equilibrium. The only other equilibrium could be for prices with \( p_2 = 0 \), but we just checked that for such equilibria agent \( b \) doesn’t trade, i.e. she consumes her full endowment (given her first and second stage beliefs). The argument in step 2 below also shows that agents \( a, c \) demand positive quantities of good 2, so that there is excess demand in the good 2 market when \( p_2 = 0 \).

**Step 2:** There are no other equilibrium outcomes in proper LMP-SEU economies.

By the preceding argument, lower priority beliefs are only relevant for boundary prices, and the only boundary price at which demands are well-defined is \( p_2 = 0, p_1, p_3 > 0 \). We verify that, for the given first-stage beliefs there is no sequence of priority-ranked beliefs for which (i) the economy is proper and (ii) there is an AD equilibrium with \( p_2 = 0 \). To show this we first compute \( D_a(p), D_a^1(p) \) (resp. \( D_b(p), D_b^1(p) \)). For this, we consider the two agent economy consisting of agents \( a, c \), endowments \( \omega_a = (1, 0), \omega_c = (0, 1) \), and common prior on states \( s_1, s_3 \) given by \( \pi_{s_1} = 1/2 = \pi_{s_2} \). Agents have SEU preferences over state-contingent pairs with state-independent kernels \( u_a, u_c \). Given a price \( p \) with \( p_2 = 0 \), let \( D_a^0(p), D_a^1(p) \) denote the demands of agents \( a, c \) in the corresponding two-state, two-good economy. By the same reasoning as above, both demands are singleton. Abusing notation let \( (x_a^1, x_a^3) \) (resp. \( (x_c^1, x_c^3) \)) denote this demand. Now go back to the three-state economy and let \( x_a^2 \) be such that \( u(x_a^2) = .5u(x_a^1) + .5u(x_a^3) \). Define analogous bundles for agent \( c \), i.e. \( x_c^1, x_c^3 \) is the two-state demand and \( x_c^2 \) solves the same equation as \( x_a^2 \), with \( u_c \) in place of \( u_a \). We then have: (fixing the \( p \) with \( p_2 = 0 \) and \( \Pi_1 \))

- \( D_a^1(p) = \{ x_a = (x_a^1, x_a^2, x_a^3) : x_a^2 \in [x_a^2, 1] \} \).
- \( D_c^1(p) = \{ x_c = (x_c^1, x_c^2, x_c^3) : x_c^2 \in [x_c^2, 1] \} \).

First note that if \( \Pi_k^b \) omits \( \pi = (.5, 0, .5) \) for some stage \( k \), then the preceding argument showed that there is excess demand for good 2. Moreover, if \( \Pi_k^b \) does not omit \( (.5, 0, .5) \) for any \( k \) then \( D_b(p) = D_b^1(p) \). It follows that, if there is to be a proper LMP-SEU economy with an AD equilibrium in which \( p_2 = 0 \), then we must have either \( D_a(p) \neq D_a^1(p) \) or \( D_c(p) \neq D_c^1(p) \). We next prove that the requirement that \( D_a^1(p) \neq D_a(p) \) or \( D_c^1(p) \neq D_c(p) \) implies excess demand in the good 2 market.
The argument for \( b \) is symmetric.

Recall the (common) first-stage beliefs are \( \Pi^1 := \{ \pi : \pi_1 = \pi_3 \in [0.5 - \varepsilon/2, 5] \} \). In a proper LMP-SEU economy we have \( \Pi^k_a \subseteq \Pi^1 \) (for the \( k \)-th stage beliefs for agent \( a \)). Let \( k \) be the first step in the chain where some tie is broken in the set \( D^1_a(p) \). Put \( \Pi^k_a = \{ \pi : \pi_1 = \pi_3 \in [0.5 - \varepsilon^1_k, 5 - \varepsilon^2_k] \} \). Note that if \( \varepsilon^1_k = 0 = \varepsilon^2_k \), then we would have \( D_a(p) = D^1_a(p) \). Hence, put \( \varepsilon^1_k > \varepsilon^2_k \geq 0 \). The assumption that some tie is broken at stage \( k \) (from the set \( D^1_a(p) \)) implies that \( \varepsilon^2_k > 0 \). But then, we must have \( D_a(p) = (x^1_a, x^3_a) \), i.e. agent \( a \) demands all of good 2. Since \( x^2_c > 0 \) for all \( x_c \in D_c(p) \) this implies there is excess demand for good 2. Hence, the only AD equilibrium possessed by a proper LMP-SEU economy is the no-uncertainty outcome. □

2.2 Axioms and Model

Let \( X \) denote a finite set and put \( \Delta(X) := \{ p \in \mathbb{R}^{\lvert X \rvert} : (i) p_i \geq 0, (ii) \sum p_i = 1 \} \).

Let \( S := \{ s_1, \ldots, s_k \} \) denote a finite set of objective states and put \( B(S) := \Delta(X)^S \), i.e. functions mapping from states to lotteries. We call these functions acts. For \( \ell \in \Delta(X) \) let \( \ell \) denote the act which gives lottery \( l \) in every state. The primitive is a preference relation \( \succeq \subseteq B(S) \times B(S) \). We impose the following axioms on \( \succeq \).

**Axiom 1:** (Order) \( \succeq \) is complete and transitive.

**Axiom 2:** (Archimedean Continuity) For any triple of lotteries \( p, q, r \in \Delta(X) \) such that \( \bar{p} \succ \bar{q} \succ \bar{r} \), \( \exists \beta \in (0, 1), \gamma \in (0, 1) \) such that \( \beta \cdot \bar{p} + (1 - \beta) \cdot \bar{r} \succ \bar{q} \succ \gamma \cdot \bar{p} + (1 - \gamma) \cdot \bar{r} \).

This is the standard continuity axiom used to characterize objective (i.e. von-Neumann-Morgenstern) expected utility. It is also the continuity axiom used in the Blume et al. (1991a) characterization of lexicographic SEU.

**Axiom 3:** (c-Independence) Fix any \( \ell \in \Delta(X) \). Then, \( f \succeq g \) if and only if \( \alpha \cdot f + (1 - \alpha) \cdot \ell \succeq \alpha \cdot g + (1 - \alpha) \cdot \ell, \forall \alpha \in [0, 1] \).

This axiom, which appears in Gilboa and Schmeidler (1989), should perhaps be called “strong” c-Independence and strengthens Maccheroni et al. (2006)’s “weak c-Independence”. For subsequent use, we recall the usual independence axiom (in a footnote), and denote it as “Axiom 3a”.\(^{10}\)

\(^{10}\)Axiom 3a: \( f \succeq g \Rightarrow \alpha f + (1 - \alpha) h \succeq \alpha g + (1 - \alpha) h, \forall \alpha \in [0, 1] \).
**Axiom 4**: (Monotonicity) If \( f(s) \geq g(s), \forall s \in S \), then \( f \geq g \).\(^{11}\)

**Axiom 5**: (Convexity) For each \( f \), the set \( UC(f) := \{ g \in B(S) : g \succeq f \} \) is convex.

This axiom requires that upper contour sets, for any act \( f \), are convex. This is implied by the independence axiom, so that relative to Blume et al. (1991a) we are, in a sense, breaking up independence into two separate postulates (resp.) axioms 3 and 5.\(^{12}\) We now turn to the formal definition of the LMP-SEU model.

**Definition 5** (Abstract lexicographic preference). Let \( X \) denote any domain and consider any ordered collection of functions \( F := (f_1, \ldots, f_n) \) defined on \( X \). The lexicographic preference on \( X \) with respect to \( F, \succeq_{\text{LEXI}} \), is defined as follows. For any \( x, y \in X \), let \( \theta(x, y) \) denote the first element (when defined) in the set \( \{1, 2, \ldots, n\} \) such that \( f_i(x) \neq f_i(y) \). Put:

- \( x \succ_{\text{LEXI}} y \) if \( \theta(x, y) \) is defined and \( f_i(x) = f_i(y), \forall i < \theta(x, y) \) or \( f_{\theta(x,y)}(x) > f_{\theta(x,y)}(y) \),
- \( y \succ_{\text{LEXI}} x \) if \( \theta(x, y) \) is defined and \( f_i(x) = f_i(y), \forall i < \theta(x, y) \) or \( f_{\theta(x,y)}(x) < f_{\theta(x,y)}(y) \), and
- \( x \sim_{\text{LEXI}} y \) if \( \theta(x, y) \) is undefined.

We will apply this concept with \( X = B(S) \) and \( F \) equal to an ordered collection of max-min expected utility functions on \( B(S) \).

**Definition 6** (LMP-SEU representation). Fix a vNM utility \( u : \Delta(X) \to \mathbb{R} \) and a collection of probabilities, \( \{\Pi_i\}_{i=1}^n \), where each set \( \Pi_i \) is a closed and convex subset of \( \Delta(S) \). Let \( \text{MEU}_{\Pi_i}(f) = \min_{\pi \in \Pi_i} E_{\pi} u(f) \) and put \( \mathcal{F} = \{\text{MEU}_{\Pi_1}(\cdot), \ldots, \text{MEU}_{\Pi_n}(\cdot)\} \). Say that a preference \( \succeq \) on \( B(S) \) has an LMP-SEU representation by the pair \( (u, \{\Pi_i\}) \) if the lexicographic functional \( \succeq_{\text{LEXI}} \) represents \( \succeq \).

### 3 Representation theorem

Our main result is a representation theorem for the LMP-SEU criterion.

**Theorem 1.** A preference \( \succeq \) satisfies Axioms 1-5 if and only if it admits a lexicographic multiple priors (LMP-SEU) representation.

---

\(^{11}\)Here we abuse notation and let \( f(s) \) (resp. \( g(s) \)) denote the act which gives the constant lottery \( f(s) \) (\( g(s) \)) in every state.

\(^{12}\)More precisely, independence implies axiom 5 and weak \( c \)-independence.
Sketch of proof. The proof is in the appendix. Here we provide a brief sketch and intuition for our construction. An LMP-SEU representation is a pair \( (u, \{ \Pi_i \} ) \) consisting of a vNM utility \( u \) and a collection of convex sets of priors, \( \{ \Pi_i \} \). Since our axioms impose an vNM preference on constant acts, the \( u(\cdot) \) is just a cardinal representation of this preference. We recover the collection \( \{ \Pi_i \} \) in several steps. To motivate the construction, it may be useful to recall how we recover priors from preferences in a well-known example such as the Gilboa-Schmeidler multiple priors model (Gilboa and Schmeidler (1989), hereafter GS). Our axioms differ from the GS axioms only in the continuity condition, i.e. we impose archimedean continuity as opposed to requiring that upper (resp. lower) contour sets of each act are closed. This makes a difference because it prevents us from being able to assert the existence of a certainty equivalent. Lexicographic models of choice, more generally, do not generally admit certainty equivalents. To see why, consider a two-stage LMP-SEU representation, \( (u, \{ \Pi_i \}_{i=1}^2) \). For a given act \( f \), put \( c_f := u^{-1}(\min_{\pi \in \Pi_1} E_{\pi} u(f)) \) as the “first-stage” equivalent of \( f \).

More generally, \( c_f \in u^{-1}(\min_{\pi \in \Pi} E_{\pi} u(f)) \) since there will be multiple lotteries lying on each indifference plane for \( u(\cdot) \), but we obscure this distinction for this discussion.

This would be the certainty equivalent were there no second stage of choice. However, if we have \( \min_{\pi \in \Pi_2} E_{\pi} u(f) > u(c^1_f) \) then according to LMP-SEU model the DM ranks \( f \) strictly higher than its first-stage equivalent \( c_f \). Similarly, putting \( c^1_f := u^{-1}(\min_{\pi \in \Pi_2} E_{\pi} u(f)) \) as the “second-stage” equivalent it is typically not the case that \( f \sim c^1_f \) (where \( c^1_f \) is interpreted as a constant act delivering the lottery \( c^1_f \) in every state). For example, if \( c_f > c^1_f \), then under the LMP-SEU representation \( f \) is strictly preferred to the constant act \( c^1_f \). It is straightforward to see from this that the only acts which possess certainty equivalents are those for which all first, second, and higher stage equivalents are the same, i.e. all indifferent lotteries under \( u(\cdot) \).

This discussion suggests that instead of a single equivalent we need to consider a vector of contingent equivalents, e.g. \( (c_f, c^1_f, c^2_f, \ldots) \) in the above notation. Under the LMP-SEU representation we can assert indifference between acts \( f, g \) precisely when their vectors of contingent equivalents are the same. This isn’t immediately helpful in deriving the representation since the sequence of equivalents are derived objects, i.e. they are defined based on the elicited priors. However, this dependence occurs in a manner that still allows us to separately elicit priors and equivalents. Namely, we can define a candidate equivalent, \( c_f \), to be the highest ranked constant act lying just below the indifference class of \( f \). The first stage priors, i.e. the set \( \Pi_1 \), consists of measures which support the following cone:

\[
C^1_f := \{ d \cdot (u(g) - u(c_f)) : d \geq 0, c_g \geq c_f \}.
\]

Readers familiar with the MEU representation will notice that, when we have cer-
tainty equivalents, this is exactly how the priors are usually elicited, i.e. they support the indifference curve of $f$ at the constant act that is just indifferent to $f$. Once we derive the first-stage priors, we define the second-stage equivalent of $f$, via $u(c_f^1) := \min_{\pi \in \Pi_1} E_{\pi} u(f)$.$^{15}$ This is a derived quantity, but it is derived from objects that are first elicited, i.e. $c_f, \Pi_1$. Now define a second cone,

$$C_f^2 := \{ d \cdot (u(g) - u(c_f^1)) : d \geq 0, g \succeq c_f, c_g^1 \succeq c_f^1 \}.$$ 

Second-stage measures $\Pi_2$ are the set of measures supporting this cone.$^{16}$

We now extend this procedure inductively. Given the first and second-stage measures, define a third-stage equivalent, $c_f^2$, as the solution to: $u(c_f^2) = \min_{\pi \in \Pi_2} E_{\pi} u(f)$. Using this, we similarly define a cone $C_f^3$, and the third-stage measures are obtained as the supporting measures of this cone. This process can be continued ad infinitum, yielding at stage $n$, a sequence of sets of measures $\{\Pi_i\}_{i=1}^n$. Let $\succeq_n$ denote the LMP-SEU preference constructed from $(u, \{\Pi_i\}_{i=1}^n)$. We show (lemma 2) that each of these representations is a coarsening of the original preference over acts.$^{17}$ Next, we use our construction of priority-ordered certainty equivalents. Map each act $f$ to its sequence, $(c_f, c_f^1, \ldots)$, of elicited certainty equivalents. Denote the image vector of certainty equivalents as $\phi(f)$. Since the representations $\succeq_n$ coarsen the original preference, the map $\phi$ is constant on indifference classes. We show, furthermore, that it is injective on indifference classes (lemma 3). Hence, the indifference class of an act is characterized by its sequence, $\phi(f)$, of certainty equivalents as opposed to a single certainty equivalent. We conclude (lemma 4) by verifying that whenever $f \not\sim g$, the associated sequences $\{\phi(f), \phi(g)\}$ of certainty equivalents must disagree at some point $N$. This bound can be chosen to be independent of the pair $(f, g)$.$^{18}$ This then implies that the stage $N$ LMP-SEU model represents the original preference over acts, and concludes the proof sketch.

The class of LMP-SEU models formally enlarges the class of LSEU models by introducing uncertainty (ambiguity) at each level of the representation. However, this is a distinction between the models that may not necessarily appear in observable behavior. One natural question is whether, within this larger class of models, we can behaviorally distinguish LMP-SEU decision-makers who perceive uncertainty from

\begin{itemize}
\item[$^{15}$] Here $c_f^1 \in u^{-1}(\min_{\pi \in \Pi_1} E_{\pi} u(f))$.
\item[$^{16}$] More precisely, we take the union over all $c_f, c_f^1$ and define the second-stage set of measures to be the union of the supporting measures. This step is also needed for the derivation of MEU measures, and is needed in our derivation of first-stage measures as well. For an exposition of this derivation for the MEU model see, e.g. Chandrasekher (2018).
\item[$^{17}$] Denoting the stage $n$ LMP-SEU preference as $\succeq_n$, we say that $\succeq_n$ coarsens $\succeq$ if $f \succeq g \Rightarrow \neg (g \succ_n f)$.
\item[$^{18}$] Choosing a uniform $N$ the claim then is that, for any $(f, g)$, the sequences $\phi(f), \phi(g)$ disagree by time $N$ (and possibly sooner).
\end{itemize}
those that do not. An ideal answer to this question would be: an LMP-SEU representation collapses to an LSEU representation, i.e. each set \( \Pi_i \) consists of a single prior, if and only if observable preferences satisfy the LSEU axioms. This turns out to be too strong of a statement. The reason is that some priors in the lexicographic chain may be irrelevant to the representation. Hence, we need to control for these redundancies before trying to identify DM’s who don’t perceive uncertainty within our model class.

There are two ways in which the priors defining an LMP-SEU representation could be irrelevant to the representation. The first is that the chain of beliefs defining the LMP-SEU representation need not be minimal, i.e. all relevant ties may already have been broken at an earlier step in the chain, in which case the additional steps are never relevant for the representation. For an example, consider the special case where each \( \Pi_i \) is singleton and consider a three-step LMP-SEU representation, \((\pi_1, \pi_2, \pi_3)\), where \( \pi_3 \) is a convex combination of \( \pi_1, \pi_2 \). Note that \( \pi_3 \) can be deleted without changing the representation.\(^{19}\) This suggests the following restriction on the set of LMP-SEU models.

**Definition 7.** An LMP-SEU model \((u, \{\Pi_i\})\) is *non-redundant* if there is no sub-collection \(\{\Pi'_i\} \subseteq \{\Pi_i\}\) such that the LMP-SEU model \((u, \{\Pi'_i\})\) also represents the same preference.

We will need one more restriction on LMP-SEU models before stating the next result. Consider two representations with ordered collections of priors, \((\pi_1, \pi_2, \pi_3)\) and \((\pi'_1, \pi'_2, \{\pi'_3 \cup \{\alpha \pi_1 + (1 - \alpha) \pi_2\}\})\). That is, in the second LMP-SEU model we have the same two singleton sets for the first two steps in the chain, but in the third set we have two priors, \(\pi_3\) and \(\alpha \pi_1 + (1 - \alpha) \pi_2\). One can show that the prior \(\alpha \pi_1 + (1 - \alpha) \pi_2\) is not relevant for the representation, i.e. we can delete this prior from the group and maintain the same preference over acts.

The argument is as follows. Put \(\pi_\alpha := \alpha \pi_1 + (1 - \alpha) \pi_2\). Consider any pair \((f, g)\) for which the third set of priors is used to break the tie and put \(f \succ g\) (the forthcoming argument applies, by symmetry, if \(g \succ f\)). We claim that the ranking MEU\(_{\Pi_3}(f) > MEU_{\Pi_3}(g)\) is obtained on the measure \(\pi_3\). If not, then it is (strictly) obtained on different measures. If the MEU value of \(f\) is attained, say, on \(\pi_3\) and the MEU value of \(g\) on \(\pi_\alpha\), then by strictness

\[
E_{\pi_3}u(g) > E_{\pi_\alpha}u(g) \quad \text{and} \quad E_{\pi_\alpha}u(f) > E_{\pi_3}u(f).
\]

Since \(E_{\pi_\alpha}u(f) = E_{\pi_3}u(g)\) (as \(E_{\pi_3}u(f) = E_{\pi_1}u(g)\) and \(E_{\pi_2}u(f) = E_{\pi_2}u(g)\)), we obtain \(E_{\pi_3}u(g) > E_{\pi_3}u(f)\) — contradicting that \(f \succ g\). Similarly, if the value of \(f\)

\(^{19}\)Reason: The measure is only relevant if it breaks a tie between \((f, g)\). But \(E_{\pi_i}u(f) = E_{\pi_i}u(g), i = 1, 2\) implies \(E_{\pi_3}u(f) = E_{\pi_3}u(g)\) since \(\pi_3 \in \text{con}(\pi_1, \pi_2)\).
is strictly obtained on \( \pi_\alpha \) and the value of \( g \) strictly on \( \pi_3 \), then we would have:

\[ E_{\pi_\alpha} u(g) > E_{\pi_3} u(g) \text{ and } E_{\pi_3} u(f) > E_{\pi_\alpha} u(f) \].

Since \( E_{\pi_\alpha} u(f) = E_{\pi_\alpha} u(g) \) we obtain

\[ E_{\pi_3} u(f) > E_{\pi_3} u(g) \] – contradicting that the ranking \( f \succ g \) requires both measures. Hence, either way we can delete the measure \( \pi_\alpha \) from the third set and maintain a representation of the same preference. This suggests the following requirement.

**Definition 8.** An LMP-SEU model \((u, \{\Pi_i\})\) is *minimal* if there is no other LMP-SEU model \((u, \{\Pi'_i\})\) such that (i) \( \Pi_i \supseteq \Pi'_i \) (where the lexicographic orders match) and (ii) \((u, \{\Pi'_i\})\) represents the same preference over acts.

When an LMP-SEU model is non-redundant and minimal we call it *regular*. We now use these definitions to show that when we strengthen convexity to independence, we recover the lexicographic SEU model of Blume et al. (1991a). The axiomatization given below is (slightly) distinct from the BBD axiomatization. We make two changes relative to the BBD axioms. First, we replace independence with convexity, and second, we add (strong) \( c \)-independence. Had we replaced strong \( c \)-independence with just \( c \)-independence the set of axioms would clearly collapse to the BBD set since both are implied by independence.

The stronger set of axioms is needed since we are offering a slightly different result: once we eliminate redundancies in the LMP-SEU representation any LMP-SEU model collapses to lexicographic SEU. Since the class of LMP-SEU models is larger than the class of lexicographic SEU models, on the expanded domain of models it might, in principle, be possible to allow a LMP-SEU representation with multiple priors. The issue is to show that any “extra” priors in such a representation must be irrelevant, so that elimination of these collapses the LMP-SEU model to lexicographic SEU.

**Corollary 1 (Lexicographic SEU).** A preference \( \succeq \) satisfies Axioms 1, 2, 3, 4, and 3a (independence) if and only if it admits a LMP-SEU representation, \((u, \{\Pi_i\})\), where each set \( \Pi_i \) is singleton. Moreover, any regular LMP-SEU model which represents \( \succeq \) has \( \Pi_i \) a singleton for each \( i \).

The proof is in the appendix. A technical contribution of our representation result is that it develops a method to construct lexicographic representations when contour sets are not necessarily straight lines. The corollary illustrates the applicability of this method since it shows how it can be applied to derive some classical results in the decision theory literature, to which the original linear methods of Hausner (1954) apply, as well as to results in which Hausner’s method does not apply (theorem 1).

## 4 Conclusion

This paper has introduced and provided axiomatic foundations for lexicographic multiple priors subjective expected utility (LMP-SEU) theory. The model gener-
alizes lexicographic subjective expected utility theory (LSEU) and max-min expected utility theory (MEU). In MEU theory, rather than a single prior over states a decision-maker has multiple priors, i.e. the decision-maker (DM) perceives uncertainty (or, ambiguity). Moreover, the DM is averse to its presence and evaluates prospects according to their lowest expected value from amongst the possible priors. In LSEU theory, there are also multiple priors but these are rank ordered, i.e. a lower ranked prior is only used to make choices when higher ranked priors cannot offer guidance. The LMP-SEU model brings together the three features in these two models, (i) the presence of ambiguity, (ii) ambiguity aversion, (iii) a rank ordering on priors.

We use this model as a prediction tool in two applications. The first is a signaling game played between an incumbent firm and a potential entrant. Here uncertainty arises from the possibility that the entrant’s toughness, e.g. her ability to withstand a price war, is unknown to the incumbent. The second example considers trade between risk-averse agents facing endowment risk. In this example the uncertainty regards the distribution of the endowment shock. Both examples share the feature that there is a unique equilibrium without uncertainty, but multiple equilibria when there is uncertainty and agents have MEU preferences. Moreover, in both examples, by applying a refinement of the LMP-SEU model we recover uniqueness of equilibrium.

5 Appendix

5.1 Proof for Section 2

Characterization of wPBE in Example 1.

Step 1: Characterization of equilibria in which no senders mix.

We need to characterize the wPBE, and show that they agree with the listed equilibria (1-3) in the text, when players have MEU preferences. We first verify there are no separating equilibria. If there were, then the signal reveals the sender’s type. Hence, the receiver would not fight following the signal sent by the strong sender and fight after the weak sender’s signal. But the weak sender would then deviate and pool with the strong sender’s signal. We check below that neither type of sender can be mixing in equilibrium. Hence, both sender types are playing pure strategies. The argument in the text already characterized the three equilibria that exist when both sender types pool on invest. Next, we verify that there are no wPBE in which both sender types don’t invest. Again, the argument in the text shows that the receiver’s off-path best-response can only possibly be to (i) mix 50:50 on fight/not fight, (ii)
to fight, or (iii) to not fight. All three cases lead to some type of sender deviating: If the receiver mixes 50:50, then the strong type of sender will deviate from “don’t invest”, unless the receiver fights with probability 1 (upon observing $di$). However, in that case, the weak type of sender will then deviate and invest. If the receiver fights upon observing $i$, the strong type deviates. Finally, note that on-path receiver behavior also must be one of (i)-(iii) (again, by the argument in example 1). Hence, if the receiver accommodates after observing $i$, then the weak type sender will deviate unless the receiver also accommodates on-path (i.e. after observing $di$). But, in this case, the strong type of sender will deviate to $i$. Hence, other than the three wPBE listed in the text there are no others in which the sender types play a pure strategy.

**Step 2:** There are no equilibria in which either sender mixes.

First, consider putative equilibria in which one mixes and the other type plays a pure strategy. If the strong type is mixing, then the receiver accommodates at the unique investment choice which reveals the sender type. If this is $di$, then the receiver deviates to $di$. Hence, the only possibility is for the weak sender to be playing $di$ with probability 1. However, for all small $\varepsilon$ any such strategy induces a strict best-response of fight when the receiver observes $di$. This induces the weak sender to deviate to $i$. If the weak type is the one mixing, then the receiver fights at the unique investment choice which reveals the sender type. If this choice is $i$, then the strong sender deviates to $i$. Hence, the only possibility is that the strong sender invests with probability 1 and the weak type mixes. For all small $\varepsilon$ the best response of the receiver is to accommodate when she observes $i$. This then induces the weak type to deviate to $i$ with probability 1. Hence, we reduce to verifying that there are no wPBE where both types mix.

Let $p_i, p_{di}$ be probabilities that the receiver fights upon observing $i$ (resp. $di$) To induce indifference by both sender types between invest/not investing the following pair of inequalities must be satisfied:

i. $4p_i + 3.1(1 - p_i) = 3.6p_{di} + 3(1 - p_{di})$.

ii. $2.5p_i + 2.7(1 - p_i) = 2.1p_{di} + 2.8(1 - p_{di})$.

Equivalently,

i. $0.9p_i - 0.6p_{di} = -1$.

ii. $-2p_i + 0.7p_{di} = 1$.

The unique solution satisfies: $0.7p_i + 0.1p_{di} = 0$. Thus, the system is incompatible with non-negative $p_i, p_{di}$, implying there is no equilibrium in which both sender types are
mixing.

Characterization of AD equilibria for example 2.

Choose the same price normalization as in the text, i.e. \( p_1 + p_1 + p_3 = 1 \). We verify that the only two AD equilibria are:

- **Equilibrium 1.** Allocations: \( x_a = (\pi_1, \pi_1, \pi_1), x_b = (\pi_2, \pi_2, \pi_2), x_c = (\pi_3, \pi_3, \pi_3) \); Prices: \( p := (p_1, p_2, p_3) = (\pi_1, \pi_2, \pi_3) \).

- **Equilibrium 2.** Allocations: \( x_a = (1/2, 1/2, 1/2) = x_c, x_b = (0, 0, 0) \); Prices: \( p = (1/2, 0, 1/2) \).

The argument in the text suggests that the relevant distinction is to consider equilibria with positive prices separately from equilibria with boundary prices, i.e. some price \( p_i \) is zero. The Inada conditions on the utility kernels ensure that, if an agent has positive wealth, then she will demand an interior allocation. Using this we now establish that, at any positive price, each agent’s demand bundle is a singleton and solves the following maximization problem (denote a generic agent with \( i \)):

\[
(*) \quad \max_{(x_1, x_2, x_3) \in B(p, \omega_i)} (0.5 - \epsilon/2) \cdot u_i(x_1) + \epsilon \cdot u_i(x_2) + (0.5 - \epsilon/2) \cdot u_i(x_3).
\]

To show this, we argue (as in the text) that at a solution to, say, agent \( a \)’s utility maximization problem UMP we must have

\[
(**) \quad u(x_2) \leq \frac{u(x_1) + u(x_3)}{2}.
\]

Otherwise, if \( u(x_2) > \frac{u(x_1) + u(x_3)}{2} \), then the MEU value of this bundle is obtained at the prior \((0.5, 0, 0.5)\). But then, agent \( a \) could sell a small amount of good 2 and with the proceeds buy (since all prices are positive) a small amount of goods 1, 3. This would raise expected utility (since the same prior is used to evaluate the adjusted bundle), contradicting that the original bundle is utility-maximizing. Hence, at an optimum we must have (**) hold.

Since each \( u_i \) is strictly concave and averages of strictly concave functions are also strictly concave, by (*) we obtain that each agent’s demand correspondence is obtained by maximizing a strictly concave function on a (convex and compact) budget set. It follows that demand exists, is single-valued, and solves (*) (and, additionally, is interior by Inada). Hence, each agent’s demand can be recovered as the solution to the first-order condition associated to problem (*). The argument given in the text when agents have the common prior \( \pi = (0.5 - \epsilon/2, \epsilon, 0.5 + \epsilon/2) \) applies verbatim, implying that the only equilibrium with positive prices is the one listed above. Now consider the possible equilibria with boundary prices. Note that
the only possible such equilibrium is where \( p_2 = 0 \), since demand is undefined (i.e., infinite) if either \( p_1 \) or \( p_3 \) equals zero. If \( p_2 = 0 \), then agent \( b \) has no wealth, hence cannot trade to obtain and of good 1 or good 3. The MEU value of her endowment is 0, with her demand correspondence being \( \{(0, x_2, 0) : 0 \leq x_2 \leq 1\} \). We also showed in the text that the demand correspondences for agents \( a, c \) had the property that:

\[
(***) \ u_i(x_2) \geq \frac{u_i(x_1) + u_i(x_2)}{2}.
\]

Given this, we can recover agent \( a, c \)'s demand in two steps. First, consider the markets in goods 1,3 alone and have agents allocate their respective budgets over demands for these two goods. Second, once these demands are computed pick any amount of \( x_2 \) that satisfies \((***)\). The solution to the first step of this program solves:

\[
\max_{(x_1, x_2, x_3) \in B_i(p, \omega_i)} 5 \cdot u_i(x_1) + 5 \cdot u_i(x_3).
\]

The unique, feasible solution to this pair of problems (for the two agents \( a, c \)) is \( x_a = (0.5, x_2^a, 5), x_c = (0.5, x_2^c, 5) \). Given that these are the demands for goods 1, 2, the unique pair \((x_2^a, x_2^b)\) that solves \((***)\) is \( x_2^a = .5 = x_2^b \). Hence, the only equilibrium in boundary prices is the listed equilibrium (equilibrium 2).

### 5.2 Proofs for Section 3

**Proof of Theorem 2.** Necessity is obvious. For sufficiency we first elicit the chain of sets of priors, \( \{\Pi_i\} \). We use these to inductively define a sequence of “pseudo”-representations which consist of lexicographically ordered sequences of MEU functionals. We then show that the sequence of functionals has finite length, which yields the desired representation. For brevity, we will abuse notation and use a “\( c \)” to denote both a generic lottery and, depending on the context, a generic constant act (paying off the lottery \( c \) in each state).

**Step 1:** Construction of first and second-stage measures.

We now extract the first element in the sequence of MEU functionals. Put

\[
c_f := \sup \{c : f > c, \exists \epsilon > 0, \forall h \in B_c(f), h > c\}.
\]

Consider for each constant act \( c \), the set \( UC^0(c) = \{f \in B(S) : c_f \geq c\} \). Note that \( UC^0(c) \) is closed and convex. Consider the restricted cone,

\[
C_c := \bigcup_{d \geq 0} d \cdot \{u(f) - u(c) : f \in UC^0(c)\}.
\]

This is a closed and convex cone. Let \( \Pi_c \) denote the supporting set of measures (positive, by monotonicity). Note that this gives a partial Bewley-style (see Bewley...
Consider, now, the corresponding MEU functional, denoted with the pair \((u, \Pi_c)\). Note that, if \(c_f > c\), then \(\min_{\pi \in \Pi_c} E_\pi u(f) \geq u(c)\). Hence, taking \(c := c_f\) we get \(\min_{\pi \in \Pi_c} E_\pi u(f) \geq u(c_f)\). For the reverse, say (towards contradiction) that \(\min_{\pi \in \Pi_c} E_\pi u(f) > u(c_f)\). Put \(\hat{c} := \min_{\pi \in \Pi_c} E_\pi u(f)\) and find \(\alpha \in (0,1), c\) such that \(\alpha \cdot \hat{c} + (1 - \alpha) \cdot c \sim c_f\). Let \(f_\alpha := \alpha \cdot f + (1 - \alpha) \cdot c\). Note that (see lemma 1) \(c_{f_\alpha} = \alpha \cdot c_f + (1 - \alpha) \cdot c < c_f\). Hence, we must have \(c_f > f_\alpha\). By the preceding argument, this can only happen if \(u(c) < u(c_f)\) or \(u(c) > u(c_f)\). Say that \(u(c) < u(c_f)\). Then, \(E_\pi u(f) \geq u(c), \forall \pi \in \Pi_c\). As before, put \(u(\hat{c}) = \min_{\pi \in \Pi_c} E_\pi u(f)\). Find \(\alpha, \hat{c}\) such that,

\(\alpha \cdot u(\hat{c}) + (1 - \alpha) \cdot u(\hat{c}) < u(c)\),

\(\alpha \cdot u(c_f) + (1 - \alpha) \cdot u(\hat{c}) > u(c)\).

Put \(f_\alpha := \alpha \cdot f + (1 - \alpha) \cdot \hat{c}\) and note that \(c_{f_\alpha} > c\). Hence, \(f_\alpha > c\), which implies \(E_\pi u(f_\alpha) \geq u(c), \forall \pi \in \Pi_c\). On the other hand, letting \(\pi^*\) be a measure on which \(\min_{\pi \in \Pi_c} E_\pi u(f_\alpha)\) is attained (as \(\Pi_c\) is compact), we find that \(E_{\pi^*} u(f_\alpha) = \alpha \cdot u(\hat{c}) + (1 - \alpha) \cdot u(\hat{c}) < u(c)\) – a contradiction. Hence, the functional \((u, \Pi)\) has the property that, for each \(f\), \(\min_{\pi \in \Pi} E_\pi u(f) = u(c_f)\). Therefore, refer to the quantity \(c_f\) as a “first-order” certainty equivalent. The pair \((u, \Pi)\) will form the first step of the lexicographic ordered chain of MEU functionals that defines the \(\gamma\) rank of an act \(f\).

\(^{20}\)The closure is w.r.t the weak* topology on \(\Delta(S)\), which coincides with the Euclidean topology in our context.
For the next step, put:

\[
UC_1^1(f) := \{g : c_g \sim c_f, \; g \succeq f\}.
\]

Notice that \(UC_1^1(f)\) picks out the subset of elements whose first-stage equivalent is in the same indifference class as that of \(f\). Using this set we form the cone,

\[
C_j^1 := \{d \cdot (u(g) - u(f)) : d \geq 0, g \in UC_1^1(f)\}.
\]

This set is obviously a cone. We claim it is also convex. Let \(\hat{g}\) be in the convex hull of the set and assume \(\hat{g} = \sum_i \alpha_i d_i g_i, d_i g_i \in \{d \cdot (u(g) - u(f)) : d \geq 0, g \in UC_1^1(f)\}\) (where \(d_i \in \mathbb{R}_+\) and \(g_i \in UC_1^1(f)\), and we wlog assume \(u(f) = 0\)). Put \(d := \sum_i \alpha_i d_i\) and note that, by axiom 5, \(\hat{g} = \hat{d} \sum_i \alpha_i d_i g_i \in UC(f)\). By the lemma below, we also have \(c_{\hat{g}/\hat{d}} = \sum_i \frac{\alpha_i d_i}{\hat{d}} c_{g_i} \sim c_f\), since each \(c_{g_i} \sim c_f\) (as \(g_i \in UC_1^1(f)\)). It follows that \(\hat{g}/\hat{d} \in UC_1^1(f)\), implying the cone \(C_j^1\) is convex.

**Lemma 1.** \(c_{\alpha \cdot g + (1 - \alpha) \cdot h} = \alpha c_g + (1 - \alpha)c_h\)

**Proof of Lemma 1.** Wlog when \(c_g \sim c_f\), we will replace ‘\(~\)’ with equality of these constants. Since \(c_{g_1} = c_{g_2}\) we obtain (by axiom 5) \(\alpha \cdot g_1 + (1 - \alpha) \cdot g_2 \succeq \alpha \cdot (c_{g_1} - c_e) + (1 - \alpha) \cdot (c_{g_2} - c_e)\) (where \(c_e\) is notation for the constant act with coordinates equal to \(\epsilon\)). Since \(c_e\) was arbitrary, this gives \(c_{\alpha \cdot g_1 + (1 - \alpha) \cdot g_2} = \alpha c_{g_1} + (1 - \alpha)c_{g_2}\). \(\Box\)

Let \(\Pi_j^1\) denote the set of measures supporting the cone. We will only consider cones \(C_j^1\) such that \(c_f \succeq f\) (\(f\) non-constant). Put \(\Pi^1 := \cup_{f \in \Pi_1} \Pi_j^1\). Note that we do not yet know that the \(\pi\)'s comprising \(\Pi_j^1\) are probabilities. First, we check that the lexicographic MEU functional \((u, \Pi, \Pi^1)\) is a coarsening of the relation \(\succeq\). From this we will deduce that the \(\pi\)'s are probability measures.

**Step 2:** Check that the lexicographic MEU function \((u, \Pi, \Pi^1)\) coarsens \(\succeq\).

Form the pair of MEU functionals \((u, \Pi, \Pi^1)\), where \(\Pi\) is the set of “first-stage” measures derived in the preceding paragraph. This gives us a two-stage lexicographic chain of MEU functionals, where we first evaluate according to the MEU functional \((u, \Pi)\), then break ties according to \((u, \Pi^1)\). We claim this lexicographic functional (call it \(\succeq\)) is a coarsening of \(\succeq\), i.e. if \(f \succeq g\), then \(\sim (g \succeq f)\). To show this, first observe that if \(c_f \succ c_g\), then \(f \succ g\). Hence, the first stage functional is a coarsening of \(\succeq\). Consider \(c_f \sim c_g\) (and assume \(f \succeq g\)). We claim that \(\min_{\pi \in \Pi^1} E_\pi u(f) \geq \min_{\pi \in \Pi^1} E_\pi u(g)\).

We check that, on each set \(\Pi_h^1\), we have \(\min_{\pi \in \Pi_h^1} E_\pi u(f) \geq \min_{\pi \in \Pi_h^1} E_\pi u(g)\). This fact will be used in a subsequent step, so we make a separate record of it.

**Lemma 2.** If \(f \succeq g\), then for each \(\Pi_h^1 \in \Pi^1\), \(\min_{\pi \in \Pi_h^1} E_\pi u(f) \geq \min_{\pi \in \Pi_h^1} E_\pi u(g)\)
Proof of Lemma 2. Note that we can assume wlog that \( f > h \), which then implies that (wlog) \( c_f > c_h \).\(^{21}\) If \( f \geq g \) and we have a reversal \( \min_{\pi \in \Pi_h} E_\pi u(g) > \min_{\pi \in \Pi_h} E_\pi u(f) \), then letting \( \pi^* \) denote a minimizer of \( \min_{\pi \in \Pi} E_\pi u(f) \) we have \( E_{\pi^*} u(g) > E_{\pi^*} u(f) \). Since \( \pi^* \) supports the pointed cone at \( h \), there must be some act \( f' \) such that (i) \( \forall \pi (\neq \pi^*) \in \Pi_h, E_{\pi^*} u(f') \geq E_{\pi^*} u(h) \) and (ii) \( E_{\pi^*} u(f') < E_{\pi^*} u(h) \).

That is, \( \pi^* \) is necessary for determining membership of \( d(u(f') - u(h)) \) in the cone \( C_h \). Consider, for any fixed \( \alpha \), the two acts \( \alpha \cdot g + (1 - \alpha) \cdot f', \alpha \cdot f + (1 - \alpha) \cdot f' \). Choose \( \alpha \) such that the following conditions hold:

1. \( \alpha \cdot E_{\pi^*} u(g) + (1 - \alpha) \cdot E_{\pi^*} u(f') > E_{\pi^*} u(h) \),
2. \( \alpha \cdot E_{\pi^*} u(f) + (1 - \alpha) \cdot E_{\pi^*} u(f') < E_{\pi^*} u(h) \).

Such an \( \alpha \) exists precisely because we have (i) \( E_{\pi^*} u(f') < E_{\pi^*} u(h) \) and (ii) we are alleging \( E_{\pi^*} u(f) < E_{\pi^*} u(g) \). But recall now that we assumed (wlog) that \( c_f \geq c_g \). Note that (i) and the fact that \( E_{\pi^*} u(f') \geq E_{\pi^*} u(h), \forall \pi \in \Pi_h \) (and that \( g \geq h \)) implies that \( \alpha \cdot g + (1 - \alpha) \cdot f' \geq h \). Hence, \( c_{\alpha \cdot g + (1 - \alpha) \cdot f'} \geq c_h \). On the other hand, (ii) implies (**) \( c_{\alpha \cdot f + (1 - \alpha) \cdot f'} \leq c_h \). Note that by replacing \( f' \) by \( f' - \varepsilon \cdot 1 \), for \( \varepsilon \) small, we can ensure that (i), (ii) still hold, that (**) holds strictly, and that \( E_{\pi^*} u(\alpha \cdot g + (1 - \alpha) \cdot f' - u(h)) \geq 0, \forall \pi \in \Pi_h \).\(^{22}\) Hence, replacing \( f' \) with the \( \varepsilon \) shift as necessary, we obtain (using lemma 1): \( \alpha \cdot c_g + (1 - \alpha) \cdot c_f' > \alpha \cdot c_f + (1 - \alpha) \cdot c_f' \). Canceling common terms (recall that strong independence holds along constant acts) we get \( c_g > c_f \) – contradiction.

For reference in step 4, where we iterate this construction, it is useful to note here how we modify this step. Let \( \Pi_h \) be a set of measures in the \( k \)-th step of beliefs on which there is an alleged reversal of the true preference \( f \geq g \). The set \( \Pi_h \) is dual to the cone, \( C_h := \{ d(u(f) - u(h)) : d \geq 0, c_1^2 \geq c_h^2, c_2^2 \geq c_h^2, c_1^{k-1} \geq c_h^{k-1} \} \). Hence, if we take \( \pi^* \) to be a measure in \( \Pi_h \) at which the MEU-value of \( u(f) \) is attained and we have \( E_{\pi^*} u(g) > E_{\pi^*} u(f) \), find \( (\alpha, f') \) such that (i), (ii) hold as above. This implies (by definition of the cone \( C_h \)) that \( c_{\alpha \cdot g + (1 - \alpha) \cdot f'} \geq c_h \), \( \forall i \leq k - 1 \). If we have inductively shown that the finite step lexicographic functionals defined by \( u, \{ \Pi^j \}_{j \leq i} \) (for \( i \leq k - 1 \)) coarsen the preference relation \( \succeq \), then it follows that \( \alpha \cdot g + (1 - \alpha) \cdot f' \succeq h \), so that \( c_{\alpha \cdot g + (1 - \alpha) \cdot f'} \geq c_h \). Similarly, we must have (via (ii) and

\(^{21}\)By mixing with a higher rank measure \( \pi \) we can always ensure that this preference holds – when we mix both \( f \) and \( g \) with the same weight and constant, this also ensures the inequality without changing the relative ranking between \( \min_{\pi \in \Pi_h^1} E_\pi u(f) \) vs. \( \min_{\pi \in \Pi_h^1} E_\pi u(g) \).

\(^{22}\)To find an appropriately small \( \varepsilon \) proceed as follows. First, observe that by mixing both \( f \) and \( g \) with a suitable mixture of a \( u \)-maximal lottery we can ensure that \( E_{\pi^*} u(g) > E_{\pi^*} u(h), \forall \pi \in \Pi_h \).

Now find \( (\alpha, f') \) such that (for this choice of \( f, g \)) inequalities (i),(ii) hold. Let \( \varepsilon_1 \) be any small number such that (i) holds with \( f' = \varepsilon_1 \cdot 1 \) replacing \( f' \). Next, pick \( \varepsilon_2 \) to be small enough so that \( \alpha \cdot \min_{\pi \in \Pi_h} [E_{\pi^*} u(g) - E_{\pi^*} u(h)] > (1 - \alpha) \cdot \varepsilon_2 \). Since \( \Pi_h \) is compact, the LHS of this inequality is positive so that there is some \( \varepsilon_2 > 0 \) that satisfies it. Put \( \varepsilon := \min \{ \varepsilon_1, \varepsilon_2 \} \) and note that we have \( E_{\pi^*} (\alpha \cdot u(g) + (1 - \alpha) \cdot u(f') - \varepsilon \cdot 1) \geq E_{\pi^*} u(h) \) for all \( \pi \in \Pi_h \) and strictly so at \( \pi^* \).
the dual characterization of the cone $C_h$, a first $i$ such that $-\langle c_{\alpha, f + (1-\alpha).f'} \rangle \geq c_h$. Hence, we must have $c_h \geq c_{\alpha, f + (1-\alpha).f'}$. Repeating the analysis by replacing $f'$ with $f' - \varepsilon \cdot \mathbf{1}$ as needed, we obtain $c_{\alpha, g + (1-\alpha).f'} \geq c_h \geq c_{\alpha, f + (1-\alpha).f'}$. Now apply lemma 1 and we obtain $c_g \succ c_f$ as before, obtaining a contradiction.

Finally, note that the preceding arguments assumed $f, g \succeq h$. This assumption is also wlog since we can mix both acts with a constant $c$ such that the mixtures are both ranked above $h$. This changes the acts, but does not change the relative ranking of their expected values.

It now follows that the 2-step (and by inductive extension, any $k$-step chain) lexicographic ranking induced by the triple $(u, \Pi, \Pi^1)$ is a coarsening of $\succeq$.

**Step 3**: Check that the measures in $\Pi^1$ are probability measures.

We are free to fix a normalization of the measures in $\Pi^1$, so we select representations whose masses (a priori, positive or negative) on each state sum to 1 (note that the sum must be positive on account of monotonicity). Towards a contradiction, say that there is some $\pi \in \Pi^1$ which is not a probability measure. Enumerate states $S = \{s_1, \ldots, s_k\}$ and say that $\pi(s_1) < 0$. By our normalization, this implies $\sum_{i \neq 1} \pi(s_i) > 0$. Also note that, by Caratheodory’s theorem (see e.g. Rockafeller (1970)), we can take $\pi$ to be an extreme point of $\Pi^1$. Since $\Pi^1$ is closed (as a subset of $\mathbb{R}^{|S|}$) and Euclidean space is reflexive, we can find an act $f_{\pi}$ such that $E_{\pi} u(f_{\pi})$ is minimized at $\pi$, i.e. $E_{\pi} u(f_{\pi}) \leq E_{\pi'} u(f_{\pi}), \forall \pi' \in \Pi^1$. Clearly we need only consider the minima over measures that are extreme points – which (by Krein-Milman) is also a closed set. Consider first the special case where the set $\Pi^1$ has finitely many extreme points, i.e. the set of supporting measures is a polytope. Then, there is an $\varepsilon$-ball around $f_{\pi}$ such that for any $g \in B_{\varepsilon}(f_{\pi})$ the minimum $E_{\pi'} u(g)$ is also attained on $\pi$ (among all $\pi' \in \text{ext}\langle \Pi^1 \rangle \setminus \pi$). Consider $g := f_1 - \varepsilon_1, f_2 - \hat{\varepsilon}, f_3 - \hat{\varepsilon}, \ldots, f_k - \hat{\varepsilon}$, where we choose $\varepsilon_i, \hat{\varepsilon}$ such that $-\pi_1 \varepsilon_1 - (\sum_{i \neq 1} \pi_i \varepsilon) = 0$. By the fact that $\pi_1 < 0$ and $\sum_i \pi_i = 1$, we must have $\sum_{i \neq 1} \pi_i > |\pi_1|$. Hence, $\varepsilon_1 > \hat{\varepsilon}$. It follows that, for this choice of $g$, the second-stage MEU functional ranks $g, f$ the same. However, note that we have $c_f \succ c_g$. Consider an $\hat{f} := f - \hat{\varepsilon}$. Note that we also have $c_f \succ c_g$. Moreover, since $E_{\pi'} u(\hat{f}) = E_{\pi'} u(f) - \hat{\varepsilon}, \forall \pi' \in \Pi^1$. Hence, the minimum is also attained at $\pi$ for the act $\hat{f}$. But since $\pi_1 < 0$ we now have that the second stage MEU functional ranks $g$ higher than $\hat{f}$, even though $c_f \succ c_g$, so that $\hat{f} \succ g$ – in contradiction to step 2 (since the two-stage functional coarsens the ranking $\succeq$).

**Step 4**: Iterate steps 2-3 to get a finite lexicographic MEU representation.

We now iterate steps 2 and 3 to successively elicit third-order beliefs, fourth-order
beliefs, and so on. For example, for each \( f \) put

\[
UC^2(f) := \{ g : g \succeq f, \ \min_{\pi \in \Pi^1} u(g) \geq \min_{\pi \in \Pi^2} u(f) \}.
\]

For brevity, put \( c^f_i := \min_{\pi \in \Pi_1} u(f) – \) call this the “second-order” certainty equivalent. Find the measures \( \Pi^2_i \) supporting the cone, \( C^2_i := \{ d \cdot (u(g) - u(f)) : d \geq 0, g \in UC^2(f) \} \). Let \( \Pi^2 = \bigcup_{f : c^f_i \succeq f, f \neq c^f_i} \Pi^2_i \) and consider the lexicographic MEU functional \((u, \Pi, \Pi^1, \Pi^2)\). The same argument as in step 2 (which does not require the elements of the quotient space to be probability measures) shows that the functional \((u, \Pi, \Pi^1, \Pi^2)\) coarsens the preference \( \succeq \). The same argument as in step 3 then shows that the elements of \( \Pi^2 \) are probability measures. Inductively obtain a sequence of lexicographic MEU functionals, \( \{(u, \Pi^1, \ldots, \Pi^n)\}_n \), where each \((u, \Pi^1, \ldots, \Pi^n)\) coarsens the original preference \( \succeq \).

Map each act \( f \) to its sequence of higher-order certainty equivalents, \( f \mapsto \{c^f_i\} \). Note that the map \( f \mapsto \{c^f_i\} \) is not injective, since whenever \( f \sim g \) we obtain (as each lexicographic function \((u, \Pi, \ldots, \Pi^n)\) coarsens \( \succeq \)) \( c^f_i = c^g_i \). However, it is injective on indifference classes of \( \succeq \).

**Lemma 3.** The map \( f \mapsto \{c^f_i\} \) is injective on indifference classes of \( \succeq \), i.e. \( \{c^f_i\} = \{c^g_i\} \iff f \sim g \).

**Proof of Lemma 3.** Consider the sequence \( \{c^f_i\} \) of certainty equivalents associated to an AA act \( f \). Break the argument into two cases, (i) \( c^f_i \sim c^f_j \) for infinitely many \( i, j \) and (ii) \( c^f_i \not\sim c^f_j \), \( \forall i, j \) (i.e. for all large \( i, j \) the certainty equivalents lie in distinct indifference classes. Consider the argument if the sets of beliefs \( \{\Pi_i\} \) were all singletons, i.e. we had a lexicographic SEU model. In that case, we would have \( f - g \in \bigcap_i \ker(\pi_i) \), where the intersection is over all \( i \). Now consider two consecutive measures, \( \pi_i, \pi_{i+1} \) in the chain.\(^{23} \)

Note that, as linear spaces, we must have \( \dim(\ker(\pi) \cap \ker(\pi_{i+1})) < \dim(\ker(\pi_i)) \).\(^{24} \)

Since the collection \( \bigcap_{i=1}^n \ker(\pi_i) \) forms a decreasing (w.r.t. linear dimension) collection of subspaces of \( \mathbb{R}^k \), it must eventually be constant. It follows that, for all large \( i \), we have \( c^f_i \sim c^g_i \). Now amend this argument for the case of the LMP-SEU representation (where the belief sets \( \Pi_i \) are non-singleton). For each \( i \), consider the sets \( \Pi_i(f) \) (resp. \( \Pi_i(g) \)) defined via, \( \Pi_i(f) = \arg\min_{\pi \in \Pi_i} E_{\pi} u(f) \). For each \( i \), find a pair of measures \((\pi_i(f), \pi_i(g)) \in \Pi_i(f) \times \Pi_i(g) \). Note that we have:

i. \( E_{\pi_i(f)} u(g) > E_{\pi_i(g)} u(g) \).

ii. \( E_{\pi_i(g)} u(f) > E_{\pi_i(f)} u(f) \).

---

\(^{23} \)The notation \( \ker(\pi) \) denotes the kernel (zero set) of the linear functional represented by \( \pi \).

\(^{24} \)To see this, note first that \( \dim(\ker(\pi_i)/\ker(\pi_i) \cap \ker(\pi_{i+1})) = 1 \). The reason being that the dimension of the quotient is the same the dimension of the quotient \((\ker(\pi_i) + \ker(\pi_{i+1}))/\ker(\pi_i) \) if \( \ker(\pi_i) \neq \ker(\pi_{i+1}) \) the latter quotient has dimension 1 since the numerator has dimension \( k \) (i.e. the full dimension in \( \mathbb{R}^k \)) and the dimension of \( \ker(\pi_{i+1}) \) is \( k - 1 \).
For any constant $c > 0$ consider the pair of equations (in unknowns $x, y$):

\[
\begin{align*}
x E_{\pi_i(f)} u(f) + y E_{\pi_i(g)} u(f) &= c \\
x E_{\pi_i(f)} u(g) + y E_{\pi_i(g)} u(g) &= c.
\end{align*}
\]

By inequalities (i),(ii) this system is always solvable. To see this concretely, put $A(f, g) = E_{\pi_i(f)} u(f), A(g, f) = E_{\pi_i(g)} u(f), B(f, g) = E_{\pi_i(g)} u(f). B(g, f) = E_{\pi_i(g)} u(g)$ and note that we can express the solutions to the preceding pair of equations via, (let $\Delta := A(f, g) B(g, f) - A(g, f) (B(f, g))$

\[
\begin{align*}
a. & \quad x = c \cdot (\frac{B(f, g)}{A(f, g)} - 1)/\Delta. \\
b. & \quad y = c \cdot (\frac{B(g, f)}{A(g, f)} - 1)/\Delta.
\end{align*}
\]

Note that if $\Delta > 0$, then we can scale $B(f, g), A(f, g)$ by a common constant without affecting the sign of $\Delta$ and ensuring the numerators and denominator of $x$ (resp. $y$) share the same sign. Scaling $B(f, g), A(f, g)$ by a constant amounts to scaling $f$ by some $K \cdot f$. We check that a common scaling constant $K$ can be chosen for each $i$ (since the argument fixes the pair $(f, g)$ (or $(f, K \cdot f)$) for all large $i$. To see that a common scale can be chosen, note that – by passing to a convergent subsequence if necessary – we can ensure that the pairs $(\pi_i(f), \pi_i(g)) \in \Pi_i \times \Pi_i$ converge to some $\pi(f), \pi(g)$. Hence, the proportion $E_{\pi_i(g)} u(f)/E_{\pi_i(f)} u(f)$ limits to a non-zero constant – call this $\hat{k}$. Choose $K$ such that $k \cdot K > 1$. It follows that replacing $(f, g)$ with $(Kf, g)$ we obtain $B(f, g)/A(f, g) > 1$ (for all large $i$). Hence, if $\Delta > 0$, we can ensure the solution to the system of equations is itself a positive vector. If $\Delta < 0$, then by choosing a small scale we can similarly ensure that $B(f, g)/A(f, g) < 1$ and obtain that the solution of the system is a positive vector. Fixing this pair $(Kf, g)$ we obtain that, for all large $i$, there are constants $(x^i, y^i)$ that solves the above system of equations. Replacing the $(x^i, y^i)$ (if necessary) by $(x^i/(x^i + y^i), y^i/(x^i + y^i))$ we put $\pi^i := x^i \cdot \pi_f + (1 - x^i) \cdot \pi_g$. It follows that, for all large $i$ we have $E_{\pi_i}(u(f) - u(g)) = 0$. The same argument as in the single prior case now shows that $\pi^i = \pi^j \forall i, j$. Hence, if we can ensure that – for all large $i$ – the lines co($\pi_i(f), \pi_i(g)$) are disjoint then we obtain a contradiction to the hypothesis that $c^i_f \sim c^i_g, \forall i$ (with $f \neq g$, so that $f - g$ is not a constant act).

We check that, by passing to a subsequence of $\Pi_i$’s if necessary, we can ensure the $\pi^i$’s are distinct. Pass to a subsequence such that the values $\min_{\pi \in \Pi} E_{\pi} u(f)$ (resp. $\min_{\pi \in \Pi} E_{\pi} u(g)$) are monotone. By symmetry, say that $\min_{\pi \in \Pi} E_{\pi} u(f)$ is monotone decreasing (strictly) in $i$. We inductively construct a sequence of pairs $(\pi_i(f), \pi_i(g))$ which satisfy (i),(ii) above and, additionally, have the property that $\text{co}(\pi_j(f), \pi_j(g)) \cap \text{co}(\pi_i(f), \pi_i(g)) = \emptyset, \forall (i, j) \text{ s.t. } i < j$. Since (again, by passing to a subsequence as needed) $c^i_f \not\sim c^j_g$ we can find a pair $(\pi_1(f), \pi_1(g))$ which satisfies (i),(ii). Since $\min_{\pi \in \Pi_2} E_{\pi} u(f) < \min_{\pi \in \Pi_1} E_{\pi} u(f)$, we have $\min_{\pi \in \Pi_2} E_{\pi} u(f) < \min_{\pi \in \Pi_1} E_{\pi} u(f)$,
$E_{\pi_i(f)}u(f)$. Since each of the sets $\Pi_i$ are non-singleton belief sets, we can find distinct $(\pi'_i(f), \pi'_i(g))$ such that $\min_{\pi \in \Pi_i, E_\pi u(f)} = \min_{\pi \in \Pi_i, E_\pi u(f)} $ and $\min_{\pi \in \Pi_i, E_\pi u(g)} = \min_{\pi \in \Pi_i, E_\pi u(g)}$. For each $i$, inductively choose a $\varepsilon > 0$ (there will be a dependence of the $\varepsilon$ on the set $\Pi_i$, but this is unimportant so we suppress this) small enough such that for all $\pi' \in B_\varepsilon(\pi_i(f))$ we have,

$$E_{\pi'}u(f) < E_{\pi_i(f)}u(f), \forall j < i.$$  

Having picked the $\varepsilon$-ball around $\pi_i(f)$ for each $i$, replace $\pi_i(g)$ with a mixture $\pi'_i(g) := \alpha \cdot \pi_i(g) + (1 - \alpha) \cdot \pi_i(g)$ such that $\pi'_i(g) \in B_\varepsilon(\pi_i(f))$. Notice that:

1. $E_{\pi_i(f)}u(f) < E_{\pi'_i(g)}u(f).$

2. $E_{\pi'_i(g)}u(g) < E_{\pi_i(f)}u(f).$

Moreover, note that any measure $\pi^*_i$ which is a convex combination of $\pi_i(f), \pi'_i(g)$ cannot be in the convex hull of $\pi_j(f), \pi_j(g)$ ($j < i$ and $\pi_j(f), \pi_j(g)$ having been defined by induction).\(^\text{25}\) Now apply the argument of the preceding paragraph to the pairs $\{(\pi_i(f), \pi'_i(g))\}$. For each $i$ we then obtain some $\pi^*_i \in \text{co}(\pi_i(f), \pi'_i(g))$ such that (i) $E_{\pi^*_i}u(f) = E_{\pi^*_i}u(g)$ and (ii) $\pi^*_i \neq \pi^*_j, \forall (i, j)$. This gives the desired contradiction. 

$$\square$$

Note that each $c^*_j \in [\underline{u}, \overline{u}]$, where $\underline{u}, \overline{u}$ denote the (resp.) minimal/maximal $u$-values. The lemma allows us to view $B(S)$ as a subset of $\mathbb{N}^{[\underline{u}, \overline{u}]}$, i.e. we can think of (an indifference class of) an act as an infinite sequence of numbers uniformly bounded between $\underline{u}$ and $\overline{u}$.\(^\text{26}\) For each act $f$, we now claim that there is some $N(f)$ such that whenever $n > N(f)$ either $c^*_n = c^*_j, \forall n \geq N(f)$ or $c^*_n \neq c^*_j$, for some $n < N(f)$. In other words, associating acts with strings of higher-order certainty equivalents, we obtain that the tie between a given, and fixed, $f$ is always broken (if at all) by looking at just the first $N(f)$ certainty equivalents comprising the infinite string. This fact is, of course, stronger than the preceding lemma. The reason we state it separately is that we need to use the weaker lemma to prove the stronger claim. Let $[g]$ denote the indifference class of an act $g$.

**Lemma 4.** For each $f$ there is an integer $N(f)$ such that, for any $g$ with $[g] \neq [f]$, there is some $n \leq N(f)$ with $c^*_g \neq c^*_f$.

**Proof of Lemma 4.** Fixing $(f, g)$ let $N(f, g) := \min\{n : c^*_n \neq c^*_g\}$. We need to show that there is no sequence $g_k$ with $[g_k] \neq [f]$ with $N(g_k, f) \rightarrow \infty$ (in $k$). Towards a contradiction, say that $\{g_k\}$ is such a sequence and label so that $N(g_{k_2}, f) \geq N(g_{k_1}, f)$ if $k_2 \geq k_1$. Since infinitely many of the $g_k$ have $g_k \succ f$ or $g_k \prec f$, wlog say

\(^{25}\)The reason being that $\min_{\pi \in \Pi_i, E_\pi u(f)} > E_{\pi'}u(f), \forall \pi' \in B_\varepsilon(\pi_i(f))$, by choice of $\varepsilon$.

\(^{26}\)More precisely, $\succeq$ classes of acts — though this detail will not matter for forthcoming arguments so we suppress it.
that the former holds and pass to this subsequence. Since the premise and conclusion of the lemma involve utility payoffs we will consider only the set of all utility-valued AA acts, i.e. the set \{u \circ f : f \in B(S)\}. By abuse, use the notation \(B(S)\) to denote the set of all utility-valued acts. Since \(B(S) \subseteq \mathbb{R}^{|S|}\), the span of the set \(\{g_k\}\) is finite-dimensional. Let \(g_k, k > k_n\) be a spanning set. Consider any \(g_k, k > k_n\) and express this as \(g_k = \sum_{i=1}^n \lambda^k_i g_k_i\). Let \(\lambda^+_k = \sum_{i : \lambda_i > 0} \lambda_i, \lambda^-_k = |\sum_{i : \lambda_i < 0} \lambda_i|\) and write

\[
g_k = \frac{\lambda^+_k}{\lambda^+_k} \sum_{i : \lambda_i > 0} \lambda_i g_k_i - \frac{\lambda^-_k}{\lambda^-_k} \sum_{i : \lambda_i < 0} |\lambda_i| g_k_i.
\]

Put \(\lambda(k) := \max\{\lambda^+_k, \lambda^-_k\}\) and say, wlog, that \(\lambda(k) = \lambda^+_k, \forall k\). This allows us to write

\[(*) \quad g_k = \lambda(k) \left[ \sum_{i : \lambda_i > 0} \alpha_i g_k_i - \frac{\lambda^+_k}{\lambda(k)} \sum_{i : \lambda_i < 0} \beta_i g_k_i \right], \]

where we have put \(\alpha_i := \lambda^+_i / \lambda^+_k, \beta_i := \lambda^-_i / \lambda^-_k\). Now assume, for simplicity, that for each element of the sequence, \(k_i\), there is a single common measure \(\pi^{k_i} \in \Pi^{k_i}\) at which the MEU value is attained for infinitely many elements of the set \(\{g_k\}\). And to conserve notation, say that this is the full sequence \(\{g_k\}\) itself. First pick a \(k\) with \(k < k_i\), where \(\{g_k\}_{k=1}^n\) is the spanning set. Note that passing to a large enough starting point \(\hat{k}\) of the sequence and picking a spanning set for the set \(\{g_k\}_{k \geq \hat{k}}\) we can always find such a \(k\). Take \(\pi^k\)-expected values of both sides of (*) and cancel \(E_{\pi_k^k} u(f) = E_{\pi_k^k} u(g_k)\) from both sides. This yields \(1 = \lambda(k) \cdot (1 - \lambda^+_k / \lambda(k))\), which implies: \(\lambda(k) = 1 + \lambda^-_k\). Hence, \(\sum \lambda^-_i = \sum_{i: \lambda_i > 0} \lambda^+_i + \sum_{i : \lambda_i < 0} \lambda^-_i = \lambda^+_k - \lambda^-_k = 1\). In other words, \(g_k\) is necessarily an affine combination of the vectors \(g_{k_1}, \ldots, g_{k_n}\). Note that this applies to any \(k_p > k_n\) for which \(g_{k_p}\) is expressed as a linear combination of \(\{g_k\}_{k=1}^n\). Find \(g_{k_p}\) with \(k_p > k_m\) and write \(g_{k_p} = \sum_{k=1}^m \lambda_i g_k\). By the preceding argument, this is an affine combination. Moreover, since the MEU value of the elements \(g_k\) is attained on the measure \(\pi^{k_i}\) we have, on the one hand, \(E_{\pi^{k_i}} u(g_{k_p}) = \min_{\pi \in \Pi^{k_i}} E_{\pi} u(g_{k_p}) = \min_{\pi \in \Pi^{k_i}} E_{\pi} u(f)\). Put \(c := \min_{\pi \in \Pi^{k_i}} E_{\pi} u(f)\) and note that we have, by definition of the integers \(k_i\), \(E_{\pi^{k_i}} u(g_{k_p}) > c\). Now use the (affine) decomposition, \(g_{k_p} = \lambda_1 g_{k_1} + \sum_{i \neq 1} \lambda_i g_{k_i}\). Take \(E_{\pi^{k_i}}\)-expected values of both

\[\text{Notes:}
\]

27 The maximum occurs for infinitely many \(k\) on one of the two possibilities, \(\lambda^+_k\) or \(\lambda^-_k\). If necessary pass to a subsequence on which it occurs on one of these two. Which one of the two we pick is irrelevant for the argument, hence select \(\lambda^+_k\).

28 Notice that this would necessarily be the case if we were dealing with lexicographic SEU – where there is just a single prior in each step of the chain.

29 We are also assuming, in asserting the existence of such a \(k\), that we can find a common measure \(\pi^k\) on which the MEU value is attained for such a \(k\). This hypothesis will be relaxed.
sides to get,
\[
c = \lambda_1 c + (1 - \lambda_1)c \\
= \lambda_1 E_{\pi^{k_1}} u(g_{k_1}) + \sum_{i \neq 2} \lambda_i E_{\pi^{k_i}} u(g_{k_i}) \\
= \lambda_1 E_{\pi^{k_1}} u(g_{k_1}) + (1 - \lambda_1)c.
\]
This implies, so long as \( \lambda_1 \neq 0 \), \( c = E_{\pi^{k_1}} u(g_{k_1}) \) – contradiction. \( \square \)

The lemma relied on two reductions. First, we assumed (in the preceding line) the affine combination of some \( g_{k_n} \) put non-zero weight on \( g_{k_1} \), where \( k_1 \) was the first step in the sequence defining the spanning set, \( \{g_k\} \). Call this hypothesis H1. Second, we assumed that there was a common measure \( \pi^{k_1} \in \Pi^{k_1} \) at which the MEU value of infinitely many elements in the counterexample sequence \( \{g_{k_n}\} \) was attained. Call this hypothesis H2.

**Claim 1.** Hypothesis H1 always holds.

**Proof of Claim 1.** We verify a stronger claim: for all large \( \hat{k} \) the linear span of the set \( \{g_k\}_{k \geq \hat{k}} \) has a basis, \( \{g_{k_1}, \ldots, g_{k_n}\} \), with two properties: (i) the initial element \( g_{k_1} = g_{\hat{k}} \) and (ii) there is some \( g_k' \in \{g_k\}_{k > \hat{k}} \) such that, writing \( g_k' = \sum_{i = 1}^{n} \lambda_i g_{k_i} \), we have \( \lambda_1 \neq 0 \). That we can choose a basis with the first property is obvious since, starting with any linearly independent subset we can always fill this out to a basis. Take the starting linearly independent subset to be \( g_{\hat{k}} \) itself. For property (ii), towards contradiction say that \( \hat{k} \) is such that \( \{g_k\}_{k > \hat{k}} \subseteq \text{sp}(g_{k_2}, \ldots, g_{k_n}) \). It follows that \( \dim(\text{sp}(\{g_k\}_{k > \hat{k}})) = 1 + \dim(\text{sp}(\{g_k\}_{k > \hat{k}})) \). Now iterate the argument. Find a basis, by abuse labelled \( \{g_{k_1}, \ldots, g_{k_n}\} \) (the subscripts are for brevity, they do not suggest that the bases have the same cardinality across iterations) for the span of \( \{g_k\}_{k > \hat{k}} \) with first element \( g_{k_1}, k_1 = \min\{k : g_k \in \{g_k\}_{k > \hat{k}}\} \). If property (ii) above does not hold, then we find that \( \dim(\text{sp}(\{g_k\}_{k > k_1})) = 1 + \dim(\text{sp}(\{g_k\}_{k > k_1})) \). Note that after finitely many steps, since the whole set \( \{g_k\}_{k \geq \hat{k}} \) has finite dimension, we must obtain a sub-collection \( \{g_k\}_{k > k'} \) which admits a basis \( \{g_{k_1}, \ldots, g_{k_n}\} \) where \( g_{k_1} = g_{k'} \) and \( \exists \hat{k} > k_1 \) (and \( g_{k} \in \{g_k\}_{k > k'} \)) for which \( g_k = \sum_i \lambda_i g_k_i ; \lambda_1 \neq 0 \). In fact, the latter property can be strengthened (as the preceding argument shows) to the claim that \( g_{k} = \sum_i \lambda_i g_k_i, \lambda_1 \neq 0 \) for all \( \hat{k} \) sufficiently large. \( \square \)

**Claim 2.** Hypothesis H2 is wlog.

**Proof of Claim 2.** Passing to a subsequence of \( \{g_{k_n}\} \) if necessary, the \( g_{k_i} \) obtain the value \( \min_{\pi \in \Pi^{k_i}} E_{\pi} u(g_{k_i}) \) at the same measure \( \pi^{k_i} \). The proof of lemma 3 shows that, for all large \( i \), we have \( \Pi_i(f) = \Pi_i(g_{k_i}) \) (where, recall, we have \( \Pi_i(f) = \arg \min_{\pi \in \Pi_i} E_{\pi} u(f) \)). It follows that, for each fixed \( g_{k_i} \), there is some (possibly larger index) \( N(k_i) \) such that we have not only \( \text{MEU}_{N(k_i)}(f) = \text{MEU}_{N(k_i)}(g_{k_i}) \), but additionally – the value attains on a common measure. Let \( N(k_i) \) denote the minimal
Relabel the sequence \( \{g_k\} \) with \( \{g_N(k_i)\} \); in other words, we pass to the subsequence of sets of measures, \( \{\Pi^{N(k_i)}\} \), and formally consider the same sequence of acts \( \{g_{k_i}\} \), where the re-labelling \( g_{k_i} = g_{N(k_i)} \) just accounts for the fact that we have discarded all sets of measures \( \Pi^k \) which do not lie along the subsequence \( \{N(k_i)\}_{k_i} \). Note that, on the one hand, the collection \( \{g_{N(k_i)}\} \) must be an infinite sequence if the original sequence \( \{g_{k_i}\} \) was infinite. But this, by itself, gives a contradiction. Fix a basis, and by abuse label this as \( \{g_{k_1}, \ldots, g_{k_p}\} \) (where the \( k_i \) corresponds to \( N(k_i) \)). Now pick a \( g_{N(k_j)} \) with \( k_j \gg k_p \). The argument in the proof of the lemma shows, since for all large \( N(k_i) \) all these acts obtain values on the same measure \( \pi \in \Pi^{N(k_i)} \), that \( g_{N(k_j)} \) is an affine combination of \( \{g_{k_1}, \ldots, g_{k_p}\} \). As above, express this as,

\[
(*) \quad g_{N(k_j)} = \sum_{m=1}^p \lambda_m g_{k_m}.
\]

By definition of the sequence \( N(k_j) \), since \( N(k_j) > N(k_m), m = 1, \ldots, p \), we must have: (take some \( \max_{k_m} N(k_m) < N < N(k_j) \))

i. \( \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{k_m}) = \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{k_n}), \forall m, n \in \{1, 2, \ldots, p\} \), and

ii. \( \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{N(k_j)}) \neq \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{k_m}) \).

Hence, choose a \( \pi^* \in \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{k_m}) \setminus \arg\min_{\pi \in \Pi^N} E_{\pi} u(g_{N(k_j)}) \) and take \( E_{\pi^*} \)-expected values on both sides of \( (*) \). On the LHS we obtain \( E_{\pi^*} u(g_{N(k_j)}) \neq E_{\pi^*} u(f) \). On the RHS, we have \( E_{\pi^*} u(g_{k_m}) = E_{\pi^*} u(f) \) and, moreover, by affineness \( \sum \lambda_m = 1 \). These together imply that the RHS equals \( E_{\pi^*} u(f) \), a contradiction.

□

Return now to the proof of step 4. Now consider the chain of MEU functionals, \( \{(u, \Pi^k)\}_k \). Let \( \succeq^k \) denote the preference over acts induced by the \( k \)-step chain, \( (u, \Pi^1, \ldots, \Pi^k) \). Each of these preferences is a coarsening of the original preference over acts \( \succeq \). By the lemma, for each \( (f, g) \) with \([f] \neq [g] \) we have \( f \not\succeq^N g \), where \( N \) is the upper bound found in the lemma. Since \( \succeq^N \) coarsens \( \succeq \), the tie-breaking between \( (f, g) \) is consistent with the original preference over acts. It follows that \( \succeq^N \) is a lexicographic MEU representation of \( \succeq \).

□

Proof of Corollary 1. Let \( (u, \{\Pi_i\}_{i=1}^N) \) be a lexicographic MEU functional that additionally satisfies independence. Pass to (if necessary) a regular representation. Following the notation in the previous proof let \( \{c_f^i\}_{i=1}^N \) denote the sequence of higher-order certainty equivalents associated to an act \( f \). If \( \Pi_1 \) is not singleton, take two measures, say \( \pi_f, \pi_g, \) for which there are two acts \( f, g \) whose respective MEU values agree and obtain on common measure.

\[\text{(More precisely, } N(k_i) \text{ is the first index such that, for each index thereafter the MEU values agree and obtain on common measure.}\]
for the model \((u, \Pi_1)\) are attained on the measures \(\pi_f, \pi_g\) (resp.). This means that either \(E_{\pi_f} u(f) < E_{\pi_g} u(g)\) or \(E_{\pi_g} u(g) < E_{\pi_f} u(g)\). Say wlog that the former holds and consider the mixture act, \(f \circ g = \alpha \cdot f + (1 - \alpha) \cdot g\). By mixing \(g\) with an appropriate constant act, we can assume that \(u(c_f) = u(c_g)\) and maintain the condition on the measures (viz. that the strict minimum of \(f, g\) are obtained on (resp.) \(\pi_f, \pi_g\)). Now say that we have \(f \succeq g\) and note that independence would then imply \(f \succeq f \circ g\). For small \(\alpha\) the strict minimum of \(E_\pi u(f \circ g)\) is still attained on \(\pi_f\). Thus, \(f \succeq f \circ g\) implies, on the one hand, \(E_{\pi_f} u(f) \geq E_{\pi_f} u(f \circ g)\). On the other hand, \(u(c_g) = E_{\pi_g} u(g) > E_{\pi_f} u(g)\) and \(u(c_f) = u(c_f)\), so that \(E_{\pi_f} u(f \circ g) = \alpha \cdot E_{\pi_f} u(f) + (1 - \alpha) E_{\pi_f} u(g) > u(c_f)\). Since \((u, \{\Pi_1\})\) is a representation of \(\succeq\), this implies \(f \circ g > -\) a contradiction. Hence, \(\Pi_1\) is a singleton set.

Now consider the second stage set \(\Pi_2\). We claim that regularity implies the existence of a pair of acts \((f, g)\) and, if \(\Pi_2\) is not singleton, the presence of a measure \(\pi_{f,g}\) that together satisfy the following properties:

1. \(E_{\pi_f} u(f) = E_{\pi_g} u(g)\), where \(\Pi_1 = \{\pi_1\}\),
2. \(\min_{\pi \in \Pi_2} E_\pi u(f) > \min_{\pi \in \Pi_2} E_\pi u(g)\),
3. \(\exists \pi_{f,g} \in \Pi_2\) such that (i) \(E_{\pi_{f,g}} u(g) > E_{\pi_{f,g}} u(f)\), (ii) \(\min_{\pi \in \Pi_2} E_\pi u(f) = E_{\pi_{f,g}} u(f) > \min_{\pi \in \Pi_2} E_\pi u(g)\).

To see that such a triple exists note first that there must be a pair \((f, g)\) satisfying (1), (2), else we can just delete \(\Pi_2\) from the lexicographic chain altogether and maintain a shorter representation of \(\succeq\), which would contradict regularity (namely, minimality). Let \(\Sigma_2 \subseteq B(S) \times B(S)\) denote the set of such pairs, i.e. pairs \((f, g)\) for which the second-order equivalent is used to break ties. We claim that there must be (at least) one such pair that additionally satisfies (3). Else, towards contradiction say that for every pair \((f, g) \in \Sigma_2\) we have

\[(**) E_{\pi} u(f) > E_{\pi} u(g), \forall \pi \in \Pi_2.\]

Then, consider the subset of measures,

\[\Pi'_2 = \{\pi \in \Pi_2 : \exists (f, g), E_{\pi_1} u(f) = E_{\pi_1} u(g), (f, g) \notin \Sigma_2\}.\]

These are the measures whose presence is needed in \(\Pi_2\) in order to maintain the tie between acts \(f, g\) (which remain tied based on their first-order certainty equivalent). If \((**\) holds, then we can contract \(\Pi_2\) to \(\Pi'_2\) (which is already closed) and maintain representability. By regularity, \(\Pi'_2 = \Pi_2\). However, notice that we then have (thinking of the measures \(\pi\) as linear functionals on \(R^{|S|}\) via \(\pi \mapsto E_\pi\)):

\[\ker(\pi_1) \subseteq \{u(h) : E_\pi u(h) \geq 0\}.\]
This implies that \( \pi_1 = \lambda \cdot \pi, \lambda > 0 \). Since both \( \pi_1, \pi \) are probability measures, this implies \( \pi_1 = \pi \). Hence, \( \Pi_2 = \Pi'_2 = \{ \pi_1 \} \), contradicting regularity. More generally, we claim that (by contracting the set \( \Pi_2 \) if necessary) if \( \Pi_2 \) is not singleton, then every measure in \( \Pi_2 \) is a limit of measures \( \pi \) for which there is some pair \((f, g)\) such that \( E_\pi u(f) = E_{\pi_1} u(g), f \succ g \) and \( E_\pi u(g) > E_{\pi_1} u(f) \). To see this, consider the set \( \Pi_0 := \{ \pi \in \Pi_2 : E_\pi u(f) > E_{\pi_1} u(g), \forall (f, g) \in \Sigma_2 \} \) is convex. Hence, if the set \( \Pi_2 \) is not singleton and \( \Pi_0 \) is also not singleton, we can contract \( \Pi_0 \) to any singleton element, say \( \pi_0 \), and replace \( \Pi_2 \) with the closure of \( \co(\{ \pi_0 \}) \) and still maintain representability.

Now we contract \( \Pi_2 \setminus \Pi_0 \). We break up this latter set into two (possibly empty) subsets: (for brevity, put \( \Pi_2' := \Pi_2 \setminus \Pi_0 \))

1. \( \{ \pi \in \Pi_2' : \min_{\pi' \in \Pi_2} E_{\pi'} u(f) < E_\pi u(f), \forall f \text{ s.t. } (f, g) \in \Sigma_2 \} \).

2. \( \{ \pi \in \Pi_2' : \exists (f, g) \in \Sigma_2, E_\pi u(f) > \min_{\pi' \in \Pi_2} E_{\pi'} u(g), E_\pi u(f) \leq \min_{\pi' \in \Pi_2 \setminus \{ \pi_0 \}} E_{\pi'} u(g) \} \).

That is, we break up the measures into those for which the minimum \( \min_{\pi' \in \Pi_2} u(f) \) is never attained for any act \( f \) and those for which the measure is decisive for breaking the first stage tie on some pair \((f, g)\) (for which, say, \( f \succeq g \)). Note that we can contract \( \Pi_0' \) by removing all measures in the first set without changing the representation. Hence, consider the collection of decisive measures in the sense of (2).

Say that \( \pi \) is decisive for \((f, g) \in \Sigma_2 \). This implies that either the value of \( f \) or \( g \) (on \( \Pi_2 \)) is attained on \( \pi \). Say that \( u(g) \) attains its minimum expected value on \( \pi \). If the minimum expected value of \( u(f) \) is (strictly) attained off \( \pi \), then consider the mixture \( f \circ g \) (for \( \alpha \) close to 1). The minimum of this is also (strictly) attained off of \( \pi \). Since \( \pi \) is decisive for \((f, g) \), it follows that – on any \( \pi' \) at which the minimum \( f \) is attained – we have \( E_{\pi'} u(f \circ g) > E_{\pi'} u(f) \). The minimum of \( f \circ g \) is attained on one of these \( \pi' \), so that we have \( \min_{\pi' \in \Pi_2} E_{\pi'} u(f \circ g) > \min_{\pi' \in \Pi_2} E_{\pi'} u(f) \). On the other hand, independence implies \( f \succeq f \circ g \) – a contradiction.

Hence, we reduce to the case where the minimum of \( u(f) \) and \( u(g) \) is attained on \( \pi \). Thus, to each decisive \( \pi \) we can associate a pair \((f, g) \in \Sigma_2 \) for which \( \pi \) is decisive and the values \( u(f), u(g) \) are both attained on \( \pi \). Decisiveness thus implies, for any other \( \pi' \) on which the value \( E_{\pi} u(f) \) is also attained, we must have \( E_{\pi'} u(f) \geq E_{\pi'} u(f) \). Since \( E_{\pi} u(f) > E_{\pi} u(g) \) (as \((f, g) \in \Sigma_2 \)) taking a mixture \( f \circ g \) we find that the value is strictly attained on the measure \( \pi \). For brevity, we replace \( f \) with \( f \circ g \) in what follows. Let \((f', g') \in \Sigma_2 \) be a pair such that \( E_{\pi} u(g') \geq E_{\pi} u(f') \). Then, consider the two lotteries \( f \circ f', f \circ g' \). For \( \alpha \) close to 1 the values of these lotteries are also attained on the measure \( \pi \). However, we have \( E_{\pi} u(f \circ f') = E_{\pi} u(f \circ g') \) and \( E_{\pi} u(f \circ f') \leq E_{\pi} u(f \circ g') \). Independence would require \((f \circ f', f \circ g') \in \Sigma_2 \) whenever \((f', g') \in \Sigma_2 \), a contradiction. It follows that regularity implies \( \Pi_2 = \{ \pi_2 \} \). Apply the
same reasoning iteratively to each set of measures $\Pi_k$ comprising the lexicographic chain.

References


