A Theory of Local Menu Preferences

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Abstract: Choice models which satisfy Arrow’s WARP cannot represent preferences that arise from settings in which the choice problem itself, i.e. the context, affects the decision-maker’s preferences over the objects of choice. We provide a formal definition of context-dependence and introduce a model of context-dependent preferences. Since these preferences are unobservable to the modeler, the aim of the paper is to carry out a revealed preference exercise. We offer the decision-maker choices between menus and use this to recover the context-dependent preferences which generate both choices between menus and choices from the menu. We show that context-dependence provides a source for flexibility and commitment that cannot be captured by menu choice models which take the more commonly used state-space approach. Our model provides a unified theory of preference for flexibility and commitment induced by context-dependent choice.

Keywords: Menu Choice, Preference for Flexibility, Preference for Commitment, Context Effects.

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1 Introduction

The standard approach to revealed preference uncovers a decision-maker’s (DM) preferences by subjecting him to a sequence of choice experiments. An implicit assumption that underpins this procedure is that the decision-maker comes to the choice experiment with a preconceived preference relation. In other words, it is assumed that the process of subjecting the DM to choice experiments cannot by itself alter the preferences that are revealed through these experiments. However, experimental evidence suggests that decision-makers form preferences in response to the choice problems that they are presented with.\(^1\) When preferences exhibit this dependence, we say that the decision-maker has context-dependent preferences.

When we use the term “context-dependence” we have in mind a decision-maker who develops his preferences on choices as they are offered to him. The aim of this paper is to re-examine preference for flexibility and preference for commitment from the vantage point of context-dependent preferences. We introduce a model of context-dependent reasoning in which the presence of context-dependence is itself the source for preference for flexibility and, respectively, preference for commitment. Taking choices between menus as the observable, we provide a behavioral foundation for this model. In most existing models of menu choice, the sole source of the menu preference is uncertainty about future tastes.\(^2\) This leaves out many interesting and commonplace examples of menu choice problems. Consider an example of a parent searching for a daycare for his children. There are certain attributes that he knows, perhaps from background reading or past experience, to look for in choosing a daycare, e.g., child-teacher ratio, cleanliness. However, there may be other characteristics that he wouldn’t know to consider unless he actually visits the facility in which these characteristics are present.

To illustrate, say he visits one facility with a homogenous ethnic composition of children. He thinks nothing of the composition and ranks the facility based only on child-teacher ratio and cleanliness. Next, he visits a facility that is heterogeneous with regards to ethnic composition. The contrast with the facility in which there is no diversity might prime the parent to the value of diversity, so that he now includes it in the list of attributes used to rank the facility. Hence, the parent might derive a latent “consumption utility” from diversity, but for this preference to be expressed, i.e. to recognize the value of diversity, it first requires a visit to a facility where the diversity is absent. For another example which tells a similar story, imagine a buyer choosing among insurance plans, where each insurance plan is itself a menu


\(^2\)There are some exceptions to this statement, discussed in more detail in the related literature section.
of attributes (i.e., coverage options). In this case, the buyer may have some idea of what constitutes good coverage, e.g., level of deductible, quality of the within-network healthcare providers. However, as in the daycare example some options may only enter his evaluation procedure once he actually views a plan that contains these options. For example, coverage options typically differ between independent insurance plans and managed care plans, e.g., immunizations and health screenings are more likely to be covered under the latter than the former. These are items that a buyer would likely consider useful once he is provided with a plan in which these options are covered. However, he might not think beforehand to include, say, “coverage of immunizations” on the list of criteria used to evaluate a plan until he is actually presented with a plan in which it is explicitly listed.\(^3\)

There is a common thread running through these examples. First, the DM only formulates preferences over options in the choice problem he is currently facing. That is, the support of the preference relation, e.g., features of daycare facilities, amenities in health insurance plans, etc., is restricted to the set of options in the choice problem (menu). Second, as more options are added to the menu the DM reformulates preferences over the enlarged choice domain. The resulting preference need not be an extension of the preference relation over the smaller domain. For example, in forming preferences over the enlarged domain the DM might reverse the preferences he held over pre-existing options. The combination of these two features is what we refer to as context-dependent preferences. When the DM’s preferences over choices are dependent on the choice problem, i.e., the context, he can exhibit non-trivial menu preferences.\(^4\) The source of non-trivial menu preferences in most existing menu choice models, particularly those that address preference for flexibility, is uncertainty about consumption preferences. However, in the preceding examples something other than uncertainty is the issue. The DM simply does not formulate preferences over choices which are not in the choice problem he is currently facing, e.g., he doesn’t think to include diversity in his evaluation process unless he visits a facility in which diversity is present. In evaluating each menu it is as if that menu comprises his world of options. Consequently, when we expand the menu he might alter his preferences over choices compared to the setting where these choices were in a smaller option set. These alterations – and not uncertainty – are the source of the non-trivial menu preference.

\(^3\)This illustration of “framing” is due to Ahn and Ergin (2010), where a similar example is used to suggest why a DM’s perception of uncertainty might be framed by the options presented to him. For an item like insurance, ex ante uncertainty plays an important role in shaping preferences over coverage plans. We are using a similar example to suggest that context-dependence can play a role as well.

\(^4\)What we mean by this is that the induced preference relation over menus cannot be recovered via the formula \(U(A) = \max_{x \in A} u(x)\), i.e., Arrowian choice.
The goal of this paper is to develop and provide behavioral foundations for a formal model of menu choice that matches this preference formation story. For a more focused and tractable model we narrow attention to two classes of menu preferences, preference for flexibility and preference for commitment. These are, arguably, the two types of menu preferences that have received the most attention in the axiomatic menu choice literature. Yet, there are basic examples of flexibility and commitment which cannot be explained with models in which uncertainty about consumption preferences is the source of the menu preference. Let us illustrate with two examples of menu preferences induced by contrast effects. The pair of examples serves as an archetype for the type of menu preferences we wish to model, and will be revisited at several points in the paper.

The first is an example of what we call preference for diversity. Imagine a DM is presented with two choice problems. In the first problem, he can invest in a safe (i.e. low return, low volatility) asset or a moderately risky (i.e. medium return, medium volatility) asset. In the second, he has the additional option of a very risky (i.e. high return, high volatility) asset. In the first problem the DM only holds shares of the safe asset. However, in the second problem his holdings are different. The additional investment opportunity draws his attention to the benefits of diversification and induces him to put some shares in the risky assets. Moreover, if the DM has monotone preferences (i.e. he is never worse off with more options), his welfare may be strictly higher in the second problem due to the presence of this additional opportunity. In other words, he exhibits a preference for flexibility (monotone menu preferences) even though the source of flexibility is not the typical explanation for this behavior – uncertain consumption preferences, after Kreps (1979). In this case, preferences are monotone simply on account of the DM feeling less constrained to make a choice(s) when there are several good choices available. This type of behavior has received the title preference for freedom of choice, after Sen (1991), Sen (1993), where it is argued that freedom of choice should be incorporated into welfare analysis. For brevity’s sake, we denote the same concept as “preference for diversity”.

Now consider the setting where a decision-maker expresses preference for commitment. As above, imagine the DM is offered the following two choice problems. In the first, he can invest in a moderately risky asset or a safe asset. In the second, he can invest in a moderately risky asset or a safe asset. In the second,
he is offered a choice between the safe asset, the moderately risky asset, or a highly risky asset. Suppose that in the first choice problem, the DM picks the safe asset. However, from the second choice problem, he exhibits indifference between the safe and medium-risk asset, i.e. he holds shares of both assets. Since the example is similar in spirit, but not formally identical, to the well-known “compromise effect” we call it preference for compromise. The idea is that the decision-maker wants to invest in the safe asset, but is tempted by the high return offered by risky assets. Moreover, he has a preference for commitment in the sense that he would rather be committed upfront to only make safe (i.e. riskless) investments. The choice to diversify into the middle asset is a compromise between his conflicting desires to make a safe investment and obtain a high return. Notice the contrast with preference for diversity. In that case, the addition of the high-risk asset also induces diversification into risky assets. However, in that example the option to diversify is welfare-enhancing. In the preference for commitment case, the addition of the high risk option is welfare-reducing since it induces a willingness to take on some risk, which the DM would rather have avoided ex ante.

We explain examples such as these by developing a formal model of context-dependent reasoning. The main ingredient of the model is a system of menu-indexed preferences, \( \{\succeq_A\}_{A \in \mathcal{M}} \). The symbol \( \succeq_A \) denotes a preference relation over options in the menu \( A \) and \( \mathcal{M} \) denotes the set of menus (choice problems). These relations are unobservable to the modeler, and the main exercise of the paper is to show how these relations can be inferred by offering the DM choices between menus. Context-dependence is modeled by imposing a cross-sectional restriction on the system \( \{\succeq_A\} \). To motivate this restriction, we first consider a definition of context-independence. A system of preferences \( \{\succeq_A\} \) should, intuitively, be context-independent if the DM comes to the choice problem with a preconceived ranking over all objects. The following definition, which formalizes this intuition, is how Sen (1997) defines context-independent preferences:

**Deductive Consistency**: \( (\succeq_X)_A = \succeq_A \).

In contrast, our heuristic definition of context-dependent reasoning is that the decision-maker develops his rankings over consumption choices as they are offered to him. The following condition is a formalization of this idea and constitutes our definition of context-dependence:

**Inductive Consistency**: \( A \subseteq B \) and \( x \succeq_A y \Rightarrow x \succeq_B y \).

This relaxes deductive consistency by requiring only the left-to-right (\( \Rightarrow \)) direction of the equivalence: \( x \succeq_A y \iff x \succeq_B y \) whenever \( A \subseteq B \). We think of inductive

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7See Example 1 for a concrete representation of preference for diversity.
consistency as a particular kind of preference formation rule. In words, it says that weak preferences are maintained as we enlarge the domain of the choice problem. Moreover, preferences may be coarsened, e.g. \( x \succ_A y \), yet \( x \sim_B y \).

Why might preferences be coarsened? The intuition for this, borrowed from Gilboa and Schmeidler (2003), is that choices are based on “reasons” (or, evidence), e.g. when the option set consists of the safe and medium-risk asset, the DM is insufficiently convinced to put shares in the medium-risk asset.\(^8\) However, as we add choices we might also (implicitly) add reasons that support previously spurned options. These additional reasons can then induce ties in the local relation, hence coarsening the preference. For instance, the addition of the highly volatile asset in both the flexibility and commitment settings introduces a contrast with the existing options that induces diversification into the medium-risk asset. The welfare effects of diversification in these two cases are distinct, but in both cases the reason for diversification is the same: the contrast makes the benefits of diversification more salient, viz. by drawing attention to the possibility of a high return.\(^9\)

This intuition partly supports the logic of the inductive consistency condition. However, note that the condition only allows a narrow form of preference reversal: strict preferences can become weak, but cannot be completely reversed. The reason for this is two-fold. First, it is a (weak) path-consistency requirement. Say \( x \) is chosen over \( y \) in menu \( A \), so that there is a “reason” supporting this choice in \( A \), and that \( x \) is indifferent to \( y \) in menu \( B \), suggesting that there are reasons supporting both choices. In the superset \( A \cup B \), all these reasons would evidently be present, hence consistency with these reasons would require indifference between \( x \) and \( y \). This is what we mean by (weak) path-consistency. An additional reason that we require a form of path-consistency is to obtain a tractable model. Without this property the system of preferences \( \preceq_A \) lacks formal discipline, and without some sort of discipline on the menu-indexed preferences \( \preceq_A \) the model has no empirical content. In sum, inductive consistency satisfies two requisites. First, it formalizes our heuristic that choices are only evaluated as they are offered, and that preferences are re-evaluated (but not completely reversed) as more choices are added. Second, it nests context-independent preferences (i.e. no coarsening from \( \succeq_A \) to \( \succeq_B \)) while allowing weak preference reversals. Moreover, it places constraints how these reversals propagate through the family \( \{ \preceq_A \} \) of preferences. The latter feature gives us control over the system of preferences \( \{ \preceq_A \} \), allowing an identification of the unobservable context-dependent preferences from choices over menus.

\(^8\)See Simonson (1989) for some experimental evidence that supports this intuition of choices based on reasons.

\(^9\)See Examples 1 and 2 for formal representations that model this behavior.
The weak preference reversals in the system \{\succeq_A\} are important to our theory. For instance, imagine that \(x \sim_B y\), yet \(x \succ_A y\) whenever \(A \subset B\). Either \(x \succ_B y\) or \(x \sim_B y\) would be weakly path-consistent, but the case of interest is where \(x \sim_B y\), i.e. when actual coarsening, or equivalently, a weak reversal occurs. These preference reversals create the gap between inductive and deductive consistency and form the channel through which context-dependent reasoning provides an alternative source of preference for flexibility/commitment. To illustrate, let us describe how the system of orders \{\succeq_A\} induces a preference over menus. For each choice problem (menu), the DM groups choice objects into \(\succeq_A\)-indifference classes, where objects in the same indifference class are assessed the same consumption value. Once objects are sorted into classes, the utility of a given menu is taken to be the numerical value of a top-ranked object multiplied by the size of the top tier. This yields the following expression:

\[
U(A) = |\Phi_u(A)| \times u(x, A)
\]

The value of a menu is determined by two quantities. First, the value of any of its top-ranked options (e.g. \(\max_{x \in A} u(x, A)\)). Second, by how many top-ranked options there are (e.g. \(|\Phi_u(A)|\)). The model components are described as follows:

1. \(u(\cdot, \cdot)\) is a representation of an inductively-consistent (context-dependent) system of orders \{\succeq_A\} and in (*) is evaluated at \(x \in \Phi_u(A)\).

2. \(\Phi_u(A)\) denotes the top \(\succeq_A\)-indifference class, i.e. \(\Phi_u(A) := \{x \in A : x \succeq_A y, \forall y \in A\}\).

The bracketed terms in (*) isolate two distinct behavioral phenomena. First, the term \(\Phi_u(A)\) captures the DM’s preference for freedom of choice: all else equal, it is welfare improving to have more top-ranked choices than less. Second, the utility index \(u(\cdot, \cdot)\) that represents an inductively-consistent system \{\succeq_A\} captures context-dependence. It is possible to have one of these features present without the other. For example, a context-independent freedom of choice representation might look like \(U(A) = |\Phi_u(A)| \cdot \max_{x \in A} u(x, A)\), and a model of “pure” context-dependence might look like \(U(A) = \max_{x \in A} u(x, A)\) (where the underlying \(\succeq_A\) are inductively consistent). We are interested in both behaviors and, in particular, our model has the property that the presence of context-dependence is the reason that the DM expresses

\[\text{An important comment on this notation. The key feature of the representation is the fact that the local preferences which underlie the cardinal index } u(\cdot, \cdot) \text{ are inductively consistent. This property is what gives the model its structure. In this regard, the notation } u(\cdot, \cdot) \text{ is sub-optimal since it is being used as shorthand to summarize an entire preference formation process, yet might suggest a much coarser, fully menu-dependent representation. While there is non-trivial menu-dependence in our model, the manner in which this dependence occurs is narrowly constrained by the inductive consistency condition.}\]
a preference for freedom of choice. The way this works through the representation is that the preference reversals in the system of orders \( \preceq A \) induce multiplicity of the arg max correspondence \( \Phi_u(A) \). In the preference for commitment case, this is an equivalence: the arg max correspondence \( \Phi_u(\cdot) \) is not single-valued if and only if the orders \( \{ \preceq A \} \) are inductively consistent, but not deductively consistent.\(^{11}\) In the preference for flexibility case, there are multiple sources of multiplicity – with the weak reversals of the system \( \{ \preceq A \} \) being one of them.\(^{12}\) We say more about these sources in the section where we discuss the representation. We refer to the system \( \{ \preceq A \} \) of context-dependent preferences as \textit{local menu preferences} since the menu-indexed relations, \( \succeq A \), are specific (or local) to the choice problem at hand.

The remainder of this paper is organized as follows. In section 2, we present a behavioral characterization of the representation \((u, \Phi_u)\). More precisely, we consider two families of representations – one where the function \( u(\cdot, \cdot) \) is monotone increasing in the menu argument and the other where it is monotone decreasing. The first class corresponds to the preference for flexibility setting and the second class corresponds to the preference for commitment case. Taken together, these representations provide a common rationale for behavior such as preference for diversity and preference for compromise. Section 3 presents identification results for the model and in section 4 we use this to obtain comparative statics and also compare our model of menu choice with the Kreps (1979) model. All proofs are in the appendix.

1.1 Related Literature

This paper seeks to understand framing effects in choice problems. This topic has a rich history, and has seen a revival of interest due to an infusion of new models that study novel behavioral phenomena such as flexibility, temptation, regret, shame (e.g. Kreps (1979), Gul and Pesendorfer (2001), Sarver (2008), Dekel et al. (2009), Dillenberger and Sadowski (2012)) among others. Crudely speaking, there have been two suggested approaches to analyze framing effects. Most papers take the choice domain to be a collection of framed choice problems, i.e. a set of consumption choices coupled with a frame which may affect how the DM evaluates the choices. Papers in one group, e.g. Plott (1973), Fishburn and Lavalle (1988),

\(^{11}\)This property is what we mean formally by the statement: context-dependence provides a source for preference for commitment. The terminology derives from the notion of preference for compromise, see Example 2, where an object \( y \) has the property that \( \Phi_u(\{x, y, z\}) \neq \Phi_u(\{x, y\}) \). That is, when \( z \) is added to the choice set \( y \) provides commitment value, but not otherwise. Moreover, the reason the switch occurs is that the system \( \{ \preceq A \} \) is inductively consistent, but not deductively consistent.

\(^{12}\)The other two sources are ties in the singleton order and “complements”, i.e. pairs \( (x, y) \) such that \( \{x, y\} > \{x\}, \{y\} \).
Papers in a second group take the observable to be a collection of frame-dependent preference relations on choice problems, e.g. framed Anscombe-Aumann acts in Ahn and Ergin (2010) and case-based choices in Gilboa and Schmeidler (2003) and Chambers and Hayashi (2010). Our approach is similar in purpose to these papers. Indeed, the way in which we model context-dependent choice has been inspired by Gilboa and Schmeidler (2001) and Gilboa and Schmeidler (2003). For instance, there is a close parallel between our inductive consistency condition and a key behavioral postulate in the Gilboa and Schmeidler (2003) model, the combination axiom. However, in our setting frames are menus and the observable is a preference relation on menus, whereas in Gilboa and Schmeidler (2003) the frames are cases (or, more accurately, memories) and the observable is a family of case-based preference relations.

In addition to Gilboa and Schmeidler (2003), an idea similar to inductive consistency is present is Tyson (2008). Tyson’s paper models DM’s who face cognitive constraints in evaluating choice problems. One of Tyson’s representations studies choices that are rationalized by a nested family of (possibly incomplete) relations \( \{ \succeq_A \} \). Nestedness is the right-to-left direction of the equivalence – \( [ x \succeq_A y \iff x \succeq_B y, \text{if } A \subseteq B ] \) – defining deductive consistency. We share with these two papers the notion that preferences across choice problems are assembled on a case-by-case basis, where preferences in larger problems are built-up from preferences in smaller problems. However, there are important differences as well. For instance, the only systems \( \{ \succeq_A \} \) that satisfy both Tyson’s nestedness condition and our inductive consistency condition are deductively consistent systems, i.e. standard preferences.\(^\text{13}\)

There is also axiomatic work that separately addresses preference for freedom of choice and menu effects involving preference for commitment. In addition to Sen (1991), Sen (1993) see also Puppe (1993), Bossert et al. (1994), Nehring (1996), Pattanaik and Xu (1998), and Nehring (1999). On the preference for commitment side, there is recent work in menu choice, e.g. Nehring (2006), Chatterjee and Krishna (2008), Barbos (2010), and Noor and Takeoka (2010), which extends the Gul and Pesendorfer (2001) temptation and self-control framework. These models show how self-control costs can be distorted by menu-effects, which can lead to behavior such as the compromise effect. Moreover, there are important issues particular to temptation which are addressed in these papers. Our model does not speak to these issues, so that the preference for commitment half of our theory is not a stand-alone

\(^{13}\)A more detailed comparison with the Gilboa and Schmeidler (2003) and Tyson (2008) models may be found in an online supplement to the paper: http://www.public.asu.edu/~mchandr6.
model of temptation. Its value is as a complement to the preference for flexibility case of our model. Taken together, the two halves of the theory aim to show that context-dependent reasoning can generate interesting behavior relevant to the issues of flexibility and commitment.

2 Model

The choice environment is described as follows.

- Let \( X = \{x_1, \ldots, x_n\} \) be a set with \( n \) elements (the consumption space).
- Let \( \mathcal{M} \) denote the set of all non-empty subsets of \( X \) (i.e. menus).
- Let \( P(X) := \) the set of complete and transitive binary relations on \( X \).

Henceforth, when we mention a binary relation \( \succeq \) on menus, we will mean an order on menus, i.e. an element of \( P(X) \). Given \( \succeq \in P(X) \), let \( (\succeq)|_X \) denote the induced order on singleton menus. Also let \( \sup(A) \) (resp. \( \inf(A) \)) denote the \( (\succeq)|_X \)-maximal (\( (\succeq)|_X \)-minimal) elements of \( A \). For any menu \( A \), we want to distinguish between two orders on \( A \). First, there is the (objective) order on singleton menus \( (\succeq)|_A \) (restricted to \( A \)). Second, there is the subjective order \( \succeq_A \). This denotes the local menu preference on singletons when the choice problem is \( A \). These two orders will typically disagree on some menus, unless the collection \( \{\succeq_A\} \) is deductively consistent.

We interpret menus as choice problems. Hence, by taking the primitive to be a binary relation on menus we are assuming the modeler can observe the DM’s welfare ranking (preferences) over choice problems. What we mean by this is that (i) the modeler can, in principle, offer the DM choices among menus, and (ii) the DM is formulating and (possibly) revising preferences over consumption choices along the way, as he is being offered menus to choice from. This offer process can either be a literal reference to a choice experiment or an allegory for a menu choice problem, e.g. the restaurant choice example of Kreps (1979). Hence, the (hypothetical) dataset we have in mind is exactly the same dataset that Kreps (1979) or most other menu choice papers have in mind when the primitive is taken to be a binary relation on menus. Finally, let us also comment on the use of the order axiom on menus. By our definition, a decision maker who evaluates choice problems using the formula \( U(A) = \max_{x \in A} u(x) \) is a “context-independent” DM. If we think of menus themselves as (compound) choice objects, e.g. insurance plans, restaurants, etc. then the order axiom on menus says that choices between menus are context-independent and – in particular – transitive, even though choices from menus are context-dependent. The point is that the DM can assign a value to each choice problem (and hence, have a complete, transitive “preference” over menus), while the valuation procedure
itself is a function of the menu. Again, the story to keep in mind is that the DM does not conceive of all alternative menus that could be offered in lieu of the menu that is available. This is why the valuation procedure (described via the local preferences \( \geq_A \) in our model) is dependent on the menu. In this sense, calling the primitive a “preference” is somewhat of a misnomer. It is a preference in a formal sense, but this preference is induced by the preferences over consumption choices which comprise the menu. However, these actual preferences are unobservable to the modeler. What is evidently observable is the welfare-ranking on choice problems induced by these preferences, this is the “menu preference”. In sum, the way to interpret the order axiom for this paper is that the DM can assign a number to each choice problem and the modeler has access to this ranking by offering choice experiments, a lá Krepsian restaurant choice.

This approach allows us to fully recover and “identify” the local menu preferences that are generating the choices over menus. Here we make a semantic distinction between uniqueness and identification. Our representations are not necessarily unique. Hence, by “identification” we mean that – while there may be more than one representation – we can nevertheless concretely characterize the set of all representations. To this end, we show that the set of representations has a lattice structure with a minimal element that can be explicitly constructed. Taking the minimal element as an input, we construct an algorithm to recover the full lattice of representations. Hence, by offering the DM choices among menus we can, in this sense, identify the unobservable local preferences which generate these choices. A second benefit is that by taking a common primitive with other menu choice models we can make behavioral (axiomatic) comparisons between existing models and ours.

While we share a common primitive with existing models, this paper is an atypical menu preference exercise. After Kreps (1979), choices between menus have typically been used to reveal second stage consumption preferences (when these are subject to uncertainty) and the aggregator that ties these together to generate the menu preference. In this paper, context-dependence – not uncertainty – is the source of preference for flexibility. Hence, we use choices between menus to recover the local preferences that both determine second stage choice and that generate the preferences over menus. The inductive consistency condition, which is a cross-sectional requirement on the elicited family of menu-dependent preferences, is what makes this a non-trivial revealed preference exercise.

\[ ^{14}\text{For brevity’s sake, we will revert to the phrase “menu preference” in denoting the welfare ranking on choice problems.} \]

\[ ^{15}\text{I am grateful to David Ahn for suggesting this result.} \]
2.1 Model Description

Definition 1. Given two orders $\succeq, \succeq'$ on a set $X$, we say that $\succeq'$ coarsens $\succeq$ if, for all $x, y \in X$, $x \succeq y$ implies $x \succeq' y$.\textsuperscript{16}

Definition 2. A system of orders $\{\succeq_A\}_{A \in \mathcal{M}}$ is inductively consistent if whenever $A \subseteq B$, the order $(\succeq_B)|_A$ coarsens $\succeq_A$.

The system of inductively consistent orders $\{\succeq_A\}$ is the centerpiece of the model. We will take a cardinal representation of this system to generate a numerical utility on menus, but the underlying system $\{\succeq_A\}$ is the main object of interest since it describes how the DM’s preferences evolve as he is offered menus to choose from. Having said this, if all we were interested in was identifying the preference-formation process, then it might suffice to observe choices from menus. However, we want to also understand the welfare effects of having context-dependent preferences. In other words, we want to recover both the preference-formation process and also pin down how this process generates the ranking on menus. For this reason, we study preferences over menus.

Depending on whether the menu preference exhibits preference for flexibility or preference for commitment, we will require that the collection $\{\succeq_A\}$ satisfy two of the following three properties: $(u(\cdot, \cdot))$ is a cardinal representation of the collection $\{\succeq_A\}_{A \in \mathcal{M}}$

- **Inductive Consistency**: $(\succeq_B)|_A$ coarsens $\succeq_A$ whenever $A \subseteq B$.
- **Downwards Monotonicity**: $A \subseteq B \Rightarrow u(x, A) \geq u(x, B)$.
- **Upwards Monotonicity**: $A \subseteq B \Rightarrow u(x, A) \leq u(x, B)$.

The first condition is ordinal and the latter two conditions are cardinal. A system of orders $\{\succeq_A\}_{A \in \mathcal{M}}$ satisfying the first condition is called a local menu preference. If it additionally satisfies upwards monotonicity, along with an additional technical condition on the cardinal representation $u(\cdot, \cdot)$ which we omit here, it will be called a local preference for flexibility. If it satisfies inductive consistency and downwards monotonicity, then we call it a local temptation preference. The interpretation of upwards monotonicity is that consumption utility from $x$ is higher in a larger option set because the DM is less constrained to choose $x$. For downwards monotonicity, the idea (explained in more detail later) is that the consumption utility of $x$ is lower when more options are present because $x$ represents a commitment and having more

\textsuperscript{16}In words, if $x$ is weakly preferred (i.e. either $x \succ y$ or $x \sim y$) to $y$ under $\succeq$, then $x$ is also weakly preferred to $y$ under $\succeq'$. This implies that if $x \sim y$, then $x \sim' y$. Actual coarsening occurs when $x \succ y$ and $x \sim' y$, so that some strict preferences are switched to weak preferences going from $\succeq$ to $\succeq'$. 

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options makes it more difficult to follow through on this commitment.

Let \( u(\cdot, \cdot) \) be a cardinal representation of the system \( \{ \succeq_A \} \). Denote the set of local arg-maxima via

\[
\Phi_u(A) := \{ x \in A : x \succeq_A y, \forall y \in A \}.
\]

The set \( \Phi_u(A) \) is interpreted as the set of ex post choices from the menu. It depends only on the underlying orders \( \{ \succeq_A \} \), but we maintain the “u” in the subscript for economy of notation. Given a pair \( (u, \Phi_u) \), the induced menu utility is the aggregate of the \( u \)-values of the maximizers of the local preference. The function, \( u(\cdot, \cdot) \), has no choice-theoretic meaning on values \( (x, A) \) where \( x \not\in A \), but for brevity’s sake we allow such pairs in its domain of definition. Values we assign to such pairs \( (x, A) \) have no bearing on the underlying system of local menu preferences, hence we will simply ignore them.

**Definition 3.** A function \( u : X \times M \to \mathbb{R}_{++} \) is a local preference for flexibility (LPF) representation of \( \succeq \in \mathcal{P}(X) \) if

\[
U(A) = |\Phi_u(A)| \cdot u(x, A) \quad \text{(for } x \in \Phi_u(A))
\]

represents \( \succeq \) and we additionally have:

- (Inductive Consistency) If \( A \subseteq B \) implies \( (\succeq_B) |_A \) coarsens \( \succeq_A \), where \( \{ \succeq_A \} \) is the ordinal system with cardinal representation \( u(\cdot, \cdot) \).
- (Upwards Monotonicity) If \( A \subseteq B \), then \( u(x, A) \leq u(x, B) \).
- (Technical Condition I) The function \( u(\cdot, \cdot) \) satisfies the following two conditions:
  
  i. \( u(x, A) = u(x, A \setminus y), \forall x \in \Phi_u(A), \forall y \in A \setminus \Phi_u(A) \).
  
  ii. \( u(x, A) > |X| \cdot u(y, A \setminus x) \) if \( \Phi_u(A) = \{ x \} \).

In addition to inductive consistency and upwards monotonicity, there is a third requirement – which we refer to as a technical condition since both (i)-(ii) can be thought of as growth conditions on the function \( u(\cdot, \cdot) \). Both conditions admit a straightforward interpretation. The first property formally requires that if we remove an element from the menu which was not in the arg max of the local order, then this doesn’t change the value of any of the terms that were providing flexibility. For intuition, notice that upwards monotonicity implies \( u(x, A) \geq u(x, A \setminus y) \). Thus, the requirement can be understood as ruling out \( u(x, A) > u(x, A \setminus y) \) when \( y \notin \Phi_u(A) \). In other words, when we add an option that is judged to be strictly inferior to an existing top-ranked option, then the consumption utility of the top-ranked option is unchanged by the presence of the inferior option.
The second property applies only to menus in which the top ranked indifference class is singleton, i.e., there is a unique maximum of the local order. In this case, it requires that the cardinal gaps
\[ u(x, A) - u(y, A \setminus x) \]
are sufficiently large. Note that this difference is strictly positive (invoking upwards monotonicity) since \( x \) was the unique maximizer of the local order and has now been deleted. The condition requires this difference in utility be large enough so that we can (ordinally) detect that \( x \) provides flexibility to the DM.

Finally, notice that there are three sources of multiplicity of the sets \( \Phi_u(A) \) in the LPF model. The first two are “context-independent” in that they do not depend on the reference menu. If either (i) \( \{x\} \sim \{y\} \) or (ii) \( \{x, y\} \succ \{x\}, \{y\} \), then inductive consistency implies \( x \sim_A y \) for any menu \( A \) containing \( x, y \). The third source comes from the (weak) preference reversals admitted by inductive consistency. That is, it may be that \( x \succ \{x, y\} \succ y \), yet \( x \sim_A y \) where \( x, y \in \Phi_u(A) \). Our main comparative statics result shows that, holding the other two sources constant, desire for flexibility is determined by the “level” of inductive reasoning exhibited by the local menu preferences \( \{\succeq_A\} \).

Definition 4. A non-constant function \( u : X \times \mathcal{M} \to \mathbb{R}^{++} \) is a local temptation preference (LTP) representation of \( \succeq \in \mathcal{P}(X) \) if \( U(A) = |\Phi_u(A)| \cdot u(x, A) \) (for \( x \in \Phi_u(A) \)) represents \( \succeq \) and we additionally have:

- (Inductive Consistency) \( A \subseteq B \) implies that \( (\succeq_B) |_A \) coarsens \( \succeq_A \).
- (Downwards Monotonicity) If \( A \subseteq B \), then \( u(x, A) \geq u(x, B) \).
- (Technical Condition II) Let \( K_A(z) := \frac{|\Phi_u(A \setminus z)|}{|\Phi_u(A)|} \). The function \( u(\cdot, \cdot) \) satisfies the following two conditions:
  i. \( K_A(x) \cdot u(y, A \setminus x) < u(x, A), \forall x \in \Phi_u(A) \).

17 More precisely, we will interpret one system, say \( \{\succeq_A^1\} \), of local menu preferences to exhibit more inductive reasoning than another, \( \{\succeq_A^2\} \), if \( \succeq_A^1 \) coarsens \( \succeq_A^2 \) for all menus \( A \). This means that whenever \( \{\succeq_A^1\} \) exhibits a weak reversal, then \( \{\succeq_A^2\} \) does as well. More details are in the comparative statics section.

18 Note that the constants \( K_A(z) \) depend only on the underlying ordinal data \( \succeq_A \), so that the growth conditions on the cardinal representation depend on the constants \( K_A(z) \), but not vice-versa. In other words, there is no circularity in the definition.
ii. \( u(x, A) \leq K_A(y) \cdot u(x, A \setminus y), \forall x \in \Phi_u(A), \forall y \in A \setminus \Phi_u(A). \)

In the LPF case the function \( u(\cdot, \cdot) \) is increasing in the menu argument, whereas in the LTP case it is monotone decreasing. The technical condition on the \( u(\cdot, \cdot) \) map for the LTP utility is different than in the LPF case, and not as intuitive. While the conditions only involve utility parameters, there is no obvious choice-theoretic interpretation. The condition is more complicated because LTP preferences form a richer class of preferences. For example, the DM might have a preference for commitment, e.g., an aversion to having risky assets on the menu. Yet, conditional on the presence of risky assets, he can have preference for flexibility among those choices that mitigate the temptation to accept risk. This means that he can exhibit a preference for commitment over consumption choices and, at the same time, a preference for flexibility over “consumption plans” – a distinction we will say more about shortly. Due to this added richness of the preference, the model is more difficult to pin down behaviorally.\(^{19}\)

### 2.2 Axioms

Recall that the order axiom is a standing assumption for all preferences studied in this paper. In addition to the order axiom there will be three more axioms that characterize the LPF and (resp. LTP) utility. Two of these axioms are shared between both models. To introduce these, consider a map \( C : \mathcal{M} \to \mathcal{M} \) defined by,

\[
C(A) = \begin{cases} 
\{ x \in A : A \succ A \setminus x \}, & \text{if } |A| > 1 \\
\{ x \}, & \text{if } A = \{ x \}.
\end{cases}
\]

That is, \( C(A) \) is the set of elements in \( A \) such that the DM is strictly worse off when any of these elements is deleted from \( A \). This map was introduced in Puppe (1993) and also enters in some of the results in Ergin (2003). We refer to the elements of \( C(A) \) as the “critical” elements of the menu \( A \) since – as we don’t explicitly observe second stage choice – this is an agnostic interpretation. However, to interpret axioms it is perhaps more useful to nonetheless think of \( C(A) \) as the set of second stage choices. This is how the model interprets \( C(A) \) since this set turns out to equal the local arg-max, \( \Phi_u(A) \), in the utility representation. On the other hand, since we don’t actually observe choices from menus, referring to \( C(A) \) as “choices from \( A \)” is just an interpretation.

\(^{19}\)The conceptual value (and the formal subtlety) of Theorem 2 (the representation result for this model) lies in the (axioms \( \Rightarrow \) model) direction. The result is an important part of our theory since it shows that replacing axiom \( A1^* \) with \( A1 \) yields a representation that is symmetric to the LPF model, where the flexibility is expressed over commitment plans and, at the same time, commitment is expressed over consumption choices.
To add support for this interpretation we can carry out the following exercise. Were we to observe ex post choice, say $\hat{C}(\cdot)$, one could then ask for a context-dependent rationalization of $\hat{C}(\cdot)$, i.e.

$$\hat{C}(A) = \{ x \in A : x \succeq_A y, \forall y \in A \}$$

where $\{ \succeq_A \}$ is an inductively consistent system. Mimic the necessity argument for Theorem 1 to check that if $\hat{C}(\cdot)$ has such a rationalization, then it satisfies the version of $A2$ (see below) applied to $\hat{C}(\cdot)$. In other words, $A2$ is necessary for a context-dependent rationalization of ex post choice, were we to actually observe ex post choice. Assuming only $A2$, the converse doesn’t hold. However, if we also assume the base-relation is transitive, then from choices we can construct a context-dependent rationalization. In this case, the rationalization of ex post choice mirrors the rationalization of critical points $C(\cdot)$, viz. in both cases these sets are recovered as the arg max of the local preferences. Hence, by analogy, we think of $C(\cdot)$ as the set of (implied) choices. The following two axioms, which both involve the map $C(\cdot)$, are the key behavioral conditions defining the LTP and LPF models.

**Local $C$-Monotonicity (A2):** If $A \subseteq B$ and $C(B) \cap A \neq \emptyset$, then $C(A) \subseteq C(B)$.

**Non-Emptiness (A3):** $C(A) \neq \emptyset, \forall A \in M$.

The second axiom, Non-Emptiness, appears in Puppe (1993) under the moniker freedom of choice. The set $C(A)$ is the inferred set of second-stage choices, so Non-Emptiness just says that the second-stage choice correspondence is non-empty valued. The Local $C$-Monotonicity axiom is the key behavioral restriction in the entire paper. To provide an interpretation, note that it is a strengthening of Sen’s well-known $\beta$ axiom applied to the critical points map $C(\cdot)$. We will borrow Sen’s interpretation of this axiom, as described in Kreps (1988). The axiom is described as follows, and is usually applied to an arbitrary choice correspondence $C(\cdot)$:

**Sen’s $\beta$:** If $A \subseteq B$ and $C(A) \cap C(B) \neq \emptyset$, then $C(A) \subseteq C(B)$.

The axiom is distinct from our $A2$ and this distinction is quite important formally. However, the intuition for the axioms is very related, so will provide an interpretation (based on Sen) for the weaker $\beta$-axiom. Sen’s interpretation of this

---

20 See Lemma 2 in the appendix for the formal argument.
21 Since Sen’s axiom is a weakening of $A2$ it is also implied by our representations, but it is not sufficient (if we keep the other axioms, but replace $A2$ with Sen’s $\beta$). Here is a simple example of a menu preference which satisfies order, monotonicity, and Sen’s $\beta$, but which fails $A2$, so that there is no LPF representation: $(x, z) \sim (x, y, z) \succ (x, y) \sim (y, z) \succ (y) \sim (z)$. Notice that $C(\{x, y, z\}) = \{x, z\}$ so that $C(\{x, y, z\}) \cap \{y, z\} \neq \emptyset$, yet $C(\{y, z\}) = \{y\} \notin C(\{x, y, z\})$.
axiom is as follows: *If the world champion in some game is a Pakistani, then all champions (in this game) are also world champions.* Let us elaborate on Sen’s intuition in the context of our model. Our story is that the DM treats each menu as if it comprises his world of options. The set $C(A)$ denotes the DM’s choices when his world of options is the set $A$. Now expand his world of options to $B$ and ask the question: under what conditions do his original choices remain intact when we expand his world of options to $B$? Sen’s $\beta$ says that if there is a choice when the world of options is $B$ that is also chosen in the smaller world $A$, then all choices from $A$ remain intact. The intuition for this consistency property is related to our intuition for inductive consistency, viz. choices are (implicitly) based on reasons. Thus, if there is a reason to choose $x$ over $y$ when the world is $A$, then this reason should still be part of the decision process when the world of options is expanded. Hence, the “consistency” in the expansion consistency condition refers to the reasons that are (implicitly) used to determine choice.

This axiom is the main behavioral postulate of the paper. For instance, in addition to the order axiom, our representation for preference for flexibility uses just this axiom, Non-Emptiness, and Monotonicity (on menus). The representation for preference for commitment uses the same two axioms on $C(\cdot)$ and replaces Monotonicity with a companion “commitment” axiom. Moreover, the main descriptive feature of the representation is the inductive consistency condition. Let us comment on the value added of the representation over this axiom. The main point is that the axiom only involves (inferred) choices via the map $C(\cdot)$ and from this observable we want to reconstruct the local preferences which generate these choices and the welfare ranking on menus. Inductive consistency is a condition on how the DM ranks pairs $(x, y)$, where neither $x$ nor $y$ need be choices. Local $C$-Monotonicity only lets the modeler observe whether $x$ or $y$ is chosen from a given menu, but it imposes a consistency condition on choices across menus. The value in the representation theorems is to show that this simple consistency condition on choices is enough to recover and, in a sense, identify the full family of preferences which generate these choices.

### 2.3 Preference for Flexibility

The representation of the LPF model relies on one more axiom, initially introduced by Kreps (1979) in his seminal study of preference for flexibility. We take this occasion to briefly review the Kreps model and the axioms introduced in the

 OTORH, $C(\{x, y, z\}) \cap C(\{y, z\}) = \emptyset$ – so the menu preference is consistent with $\beta$ but not with our strengthening of $\beta$. 

17
characterization. Kreps axiomatized the following utility function on menus,

\[ U(A) = \sum_{s \in S} \max_{x \in A} u_s(x) \]

where \( S \) denotes a (subjective) state space and \( \{u_s(\cdot)\}_{s \in S} \) is the set of state-dependent utility kernels. In addition to order, Kreps (1979) shows that the following axioms characterize this model:

**Monotonicity (A1⁺)**: If \( A, B \in \mathcal{M} \) and \( A \subseteq B \), then \( B \succeq A \).

**Modularity (A2⁺)**: If \( A \subseteq B \) and \( A \sim B \), then \( A \cup C \sim B \cup C, \forall C \in \mathcal{M} \).

Kreps (1979) considers an example of a DM whose period 1 problem is to choose a dinner reservation at a restaurant. Consumption takes place in period 2, but due to the uncertainty in taste preferences the DM values flexibility in period 1. Imagine that we have the ranking, \{chicken, fish\} \sim \{chicken\}. Modularity requires, \{chicken, fish, steak\} \sim \{chicken, steak\}. To interpret, note that the source of preference for flexibility in the Kreps utility comes from uncertainty about consumption preferences. In formulating a state-space, the DM accounts for all scenarios where fish could possibly provide flexibility. Hence, adding steak as an option could switch the DM’s choice to steak in some states, but it cannot alter the predetermined ranking of chicken over fish (in any state). The fact that this ranking is predetermined is the key distinction between the Kreps model and our model. In the Kreps story, the DM fully anticipates all choices he may be presented with along with all ex post preferences over these choices. Hence, there can never be an element of “surprise” when an option is added to the menu.

Now consider a different evaluation procedure. Imagine the decision-maker thumbs through a local restaurant guide and ranks menu options only if he is presented with restaurant choices that offer those options. For example, he doesn’t formulate a ranking between fish and chicken unless he is presented with a menu in which both these options are present. For each menu, he only ranks options local to that menu and, taking a cardinal representation of this ranking, attaches a value to that menu (using the LPF formula) – call this the restaurant “score”. Proceeding menu by menu, he ranks options and scores menus, possibly revising previous rankings along the way – but in a manner that is inductively consistent. Once he reaches the end of the guide, he selects the restaurant with the highest score. This is how an LPF decision-maker solves the restaurant choice problem. The value of this alternative procedure is that it provides a source of preference for flexibility that is distinct from, and not in the purview of, the Kreps (1979) model. Consider the following example.
Example 1 (Preference for Diversity). Let \( x, y, z \) respectively denote a safe (i.e. low volatility, low return) asset, a medium risk (i.e. medium volatility, medium return) asset, and a very risky (i.e. high volatility, high return) asset. Consider (a monotone extension of) the following menu preference:

\[
\{x, y\} \succ \{x\}, \{x, y, z\} \succ \{x, z\} \sim \{x\}.
\]

When the option set is \( x \) and \( y \), the investor doesn’t feel like he has much flexibility in investment options. However, when the highly risky asset is added he now feels a genuine freedom of choice in investment opportunities, perhaps because the presence of the “extreme” option \( z \) is needed to make the tradeoff between return and volatility salient. This example violates modularity, so that it cannot be explained by Kreps’ model. However, it can be explained with local menu preferences. Consider the following system of local orders:

- \( x \succ \{x, y\} \)
- \( y \succ \{y, z\} \)
- \( x \sim \{x, y, z\} \)
- \( y \sim \{x, y, z\} \)
- \( z \).

Note that the system is inductively consistent. To obtain an LPF representation we find an upwards monotonic function \( u(\cdot, \cdot) \) that represents this system, such that \( |\Phi_u(A)| u(x, A) =: U(A) \) represents the menu preference. For this example, the system of local orders that yields an LPF representation is unique up to a specification of the local order on \( \{y, z\} \). Typically, the orders \( \{\succeq_A\} \) will be non-unique but can be identified in a sense we make precise later on. However, in all cases there are many choices for the cardinal representation \( u(\cdot, \cdot) \). For example, we may take \( u(x, \{x, y\}) = 1, u(y, \{y, z\}) = 1/2, u(x, \{x, y, z\}) = u(y, \{x, y, z\}) = u(z, \{x, y, z\}) = 2 \). The following is our representation result for the LPF model.

**Theorem 1.** A preference \( \succeq \in P(X) \) satisfies \( A1^*, A2, \) and \( A3 \) if and only if it admits an LPF representation.

### 2.4 Preference for Commitment

We now turn to a utility representation for the LTP model. The main difference between this model and the LPF utility is that the index \( u(\cdot, \cdot) \) is required to satisfy Downwards Monotonicity in the LTP case. Towards an intuition for Downwards Monotonicity, we first do a brief recap of the notion of preference for commitment. To fix ideas, let \( x = \text{work}, y = \text{shirk} \). The normative preference is to work, but the DM is tempted to put off work when opportunities to shirk are available. Moreover, resisting these opportunities is costly. This might lead to the following preference on menus,

\[
\{x\} \succ \{x, y\} \succ \{y\}.
\]


This behavior is captured by the following axiom on menu preferences, introduced in Gul and Pesendorfer (2001) in their seminal piece on temptation and self-control:

**Set-Betweenness**: \( A \succeq B \Rightarrow A \succeq A \cup B \succeq B \).

Menu preferences that exhibit this property have been given the moniker “preference for commitment”. In the preference for flexibility case, objects of choice within a menu are just the singleton elements comprising the menu. However, in the preference for commitment case we interpret the objects of choice as a collection of “consumption plans”. For instance, in the preceding example, the menu \( \{x, y\} \) itself implicitly represents two plans: (i) the DM can plan to commit to work \( x \) and incur the cost to avoid shirking, or (ii) the DM can simply plan to shirk. Hence, we imagine that the DM subjectively converts a menu \( A \) into a collection of consumption plans, \( (x, A) \). We formally denote this association via,

\[
A \leftrightarrow \{(x, A) \}_{x \in A}.
\]

The pair \( (x, A) \) denotes a plan to (i) consume \( x \) and (ii) resist options in the menu \( A \) that make it costly to follow through on \( x \). In the Gul and Pesendorfer (2001) model this association occurs via the formula,

\[
U(A) = \max_{x \in A} [u(x) - \max_{y \in A} (v(y) - v(x))].
\]

If the DM evaluates menus using this formula, he converts a menu \( A \) to a set of consumption plans, each one denoted \( (x, A) \), where

\[
(x, A) := \arg \max_{y \in A} v(y) \cup \{x\}.
\]

That is, we identify the plan \( (x, A) \) with the sub-menu consisting of the worst temptations in \( A \) along with the object of commitment \( x \). Using this model we can now give a clear motivation for downwards monotonicity: As the menu enlarges, we possibly add more temptations (e.g. shirking opportunities), thereby decreasing the net value of the plan \( (x, A) \).

Interpreting consumption plans as the choice objects and elements of \( \Phi_u(A) \) as the set of consumption commitments, the LTP model tries to capture two interrelated features. First, the DM has an intrinsic preference for flexibility over plans \( (x, A) \), so that – holding temptations constant – the more commitment options there are, the better off he is. The contrast with the LPF model is in the choice domain over which the DM expresses preference for flexibility. Here it is over the (subjective) domain of consumption plans, and not over the objective domain of consumption choices. The second feature is that the DM has a non-trivial preference for commitment. The existence of this preference for commitment is expressed through the
following axiom:

**Commitment** (A1) : If $A$ is not singleton, then $A \setminus x \succeq A$ for some $x \in A$.

On a formal level, this axiom is implied by the *Set-Betweenness* axiom of Gul and Pesendorfer (2001) and, in turn, implies the *Desire for Commitment* (DFC) axiom of Dekel et al. (2009). There is also a symmetry between the Commitment axiom and the Monotonicity axiom. To see this, let us rephrase Monotonicity from the previous section:

**Monotonicity**: If $A$ is not singleton, then $A \succeq A \setminus x$ for all $x \in A$.

Say that a decision maker has a *preference for flexibility* at $A$ if Monotonicity holds at $A$. Then, saying that the DM values commitment at $A$ is nearly the logical negation of having preference for flexibility at $A$. This symmetry between the axioms is reflected in the utility representations through the monotonicity property of the local utility index, $u(\cdot, \cdot)$. The following example illustrates how context-dependence can provide a source of preference for commitment.

**Example 2** (Preference for Compromise). Let $x, y, z$ denote a safe asset, a medium risk asset, and a high risk asset. Consider the following menu preference:

$$\{x\} \sim \{x, y\}, \{x\} \succ \{x, y, z\} \succ \{x, z\}, \{x, y\} \succ \{y, z\}.$$

The DM is tempted to accept risk in exchange for the possibility of a high return, even though the normative preference is to avoid risk. To this standard preference for commitment story, we add a twist. Welfare from the menu consisting of just the safe and the medium risk option is the same as when the medium option isn’t present. Moreover, as in the preference for diversity example, the contrast created by the presence of the high risk option triggers the DM’s willingness to accept risk. However, in this case, the contrast is welfare-reducing precisely because it induces the DM to accept this risk. Thus, the medium risk option provides commitment value (as a compromise) in the presence of the high risk option, but not otherwise.

The example violates the $(A \succeq A \cup B)$ part of *Set-Betweenness*, hence it does not admit a Gul and Pesendorfer (2001) representation. It turns out that it cannot be represented by the Dekel et al. (2009) extension of the Gul and Pesendorfer (2001) extension of the DFC states: $\exists x \in A. \{x\} \succeq A$. Commitment is stronger than DFC, i.e. there are menu preferences that satisfy DFC, but not Commitment. For an example take $X = \{x, y, z\}$ and put $\{x\} \succ \{x, y, z\} \succ \{x, y\} \succ \{x, z\} \succ \{y\} \succ \{y, z\}$. The containment relations between the axioms is Lemma 1 in the appendix.
Let us construct an LTP representation of the menu preference. We first construct a system of local orders.

- $x \succ_{\{x,y\}} y$
- $y \succ_{\{y,z\}} z$
- $x \sim_{\{x,y,z\}} y \succ_{\{x,y,z\}} z$.

Now take a cardinal representation of this system, $u(\cdot,\cdot)$, that is downwards monotonic.

Put $U(A) := |\Phi_u(A)| \cdot u(x, A)$ and note that this represents the menu preference in the example. There is an ordinally unique LTP representation in this case. In general there will be more than one, however we can characterize the set of representations and show that there is always a unique system with the property that all local menu preferences that represent $\succeq$ are coarsenings of this system. This will be shown as part of the general identification result in the next section.

We now recast this example of preference for compromise in the language of a mirror example from Dekel et al. (2009) (DLR). The Dekel et al. (2009) example mildly modifies the preceding example, yet only the modification admits a state-space representation (Dekel et al. (2009) is a state-space model, à la Kreps (1979) – see the appendix for details). Both the modified example and the original (which is formally identical to Example 2 above) admit a local menu preferences representation. Let us explain the example first, then we will take a closer look at why the non-representability is occurring. The labels are taken verbatim from Dekel et al. (2009).

**Example 3** (Preference for Compromise – slight variant). Let $X = \{b, y, c\}$ and consider the following preference,

$$\{b\} \succ \{b, y\}, \{b, y, c\} \succ \{b, c\}, \{b, y, c\} \succ \{y, c\}.$$

Abbreviations are as follows: $b$ is a healthy choice (e.g. broccoli), $y$ is mildly unhealthy (e.g. frozen yogurt), and $c$ is very unhealthy (e.g. cake). The menu preference expresses the fact that yogurt is a (relative) temptation in the absence of cake, otherwise it becomes a (relative) commitment since it evidently mitigates the dessert craving when facing the menu $\{b, y, c\}$.

23 There is no axiomatization, at present, of the Dekel et al. (2009) in the discrete menus domain. However, the discrete analogue of AIC (i.e. $x \cup \{\beta\} \succ x \Rightarrow \{\alpha\} \succ x' \cup \{\beta\}$, where $\{\alpha\} \succ \{\beta\}, x' \subseteq x$) is an implication when the choice domain is the set of finite menus.

24 There are many such choices for the cardinal representation of $\{\succeq, A\}$. For example, let $u(x, \{x, y\}) = 1, u(x, \{x, y, z\}) = u(y, \{x, y, z\}) = u(y, \{x, y\}) = 1/3 = u(y, \{y, z\}), u(z, \{z\}) = 1/4$. 

22
Example 4 (Preference for Compromise, relabelled). We change yogurt from a (relative) strong temptation to a (relative) weak temptation in the menu \{b, y\}, everything else is the same as in the preceding example:

\{b\} \sim \{b, y\}, \{b, y, c\} \succ \{b, c\}, \{b, y, c\} \succ \{y, c\}.

Note that this example is formally identical to Example 2 and, moreover, both examples (3 and 4) arguably convey the same story – when temptations are added to a menu, relatively less tempting choices obtain newfound commitment value. Nevertheless, only the Example 3 admits a Dekel et al. (2009) representation. Why? To explain, we follow Dekel et al. (2009) and interpret the first example by thinking of the decision-maker as facing two ex-post states. In the first state, there is no temptation and he chooses to consume \(b\). In the second state, the self-control cost required to commit to \(b\) and stay away from \(c\) far exceeds the self-control cost required to commit to the compromise of \(y\); hence, \(y\) is consumed in this state.

While this is a compelling interpretation, Example 4 illustrates that there are subtle consequences to the state space approach. In order to explain why the DM consumes \(y\) even when \(c\) isn’t on the menu (as the DLR model requires), we imagine that the DM formulates a list of state-dependent preferences that is independent of the menu of options in front of him, i.e. by reasoning about what he would choose in the temptation state were his menu to include \(c\). Since he would consume \(y\) in this scenario, he concludes that he should then consume \(y\) even when \(c\) isn’t on the menu. However, this corrupts the explanation that \(y\) is chosen as a compromise in the menu \{b, y, c\} since there shouldn’t be any need to compromise when the menu is \{b, y\}. By focusing on a special case of the Dekel et al. (2009) representation (axiomatized in the menus of lotteries framework by Stovall (2010)), this intuition comes through clearly in the “math” – where we can explicitly show that that this example eludes a (special case of) the Dekel et al. (2009) model. The details are in the appendix under Claim 2.

Our interpretation of this example is that, as in the preference for diversity case, the source of the menu preference is context-dependence – not uncertainty. When the DM evaluates \{b, y\} he does not imagine what he would choose were his menu to include \(c\). He simply ranks the options that are in front of him, leading to the choice of the “plan” \{b\}. Once \(c\) is added to the menu, however, he exhibits indifference between the plans \{b, c\}, i.e. consume \(b\) and resist \(c\) (which tempts \(b\)), and \{y\} (consume \(y\) free of resistance). Hence, the presence of the temptation \(c\) (which lowers the net utility from consuming \(b\)) provides a reason in favor of committing to \(y\). OTOH, the original reason favoring \(b\) over \(y\), viz. \(b\) is healthy, is still present. The presence of reasons in favor of the dueling plans, \{y\} vs. \{b, y\}, induces a tie between the two plans. This is how local menu preferences interpret the example.\(^{25}\)

\(^{25}\)Noor and Takeoka (2010) observe that ex post choice in these models satisfies Arrow’s WARP
The following theorem is our representation result for the LTP model.

**Theorem 2.** A preference $\succeq \in P(X)$ satisfies $A1$, $A2$, and $A3$ if and only if it admits an LTP representation.

LTP representations exhibit the following property: The sets $\Phi_u(A)$ are non-singleton if and only if the local menu preferences $\{\succeq_A\}$ in the representation are inductively, but not deductively, consistent. Hence, the only source of freedom of choice (over commitment plans) for LTP preferences is the context-dependence of the local menu preferences $\{\succeq_A\}$. This is the content of the following corollary.

**Corollary 1.** Let $\succeq \in P(X)$ admit an LTP representation $(u, \Phi_u)$. Then, the set $\Phi_u(\cdot)$ is singleton (for all menus) if and only if the system $\{\succeq_A\}$ underlying $u(\cdot, \cdot)$ is deductively consistent.\(^{26}\)

### 3 Identification

In this section, we suppress mention of the cardinal index $u(\cdot, \cdot)$ that supports a given system $\{\succeq_A\}$ as a representation of $\succeq$. Hence, when we say that “$\{\succeq_A\}$ represents $\succeq$” what we mean is that there is a cardinal index that represents $\{\succeq_A\}$ and which generates $\succeq$ via the LPF (resp. LTP) formula. Beyond a mild result about the monotonicity property of the index $u(\cdot, \cdot)$ (part (1) of the identification theorem), we cannot identify the intensities $u(\cdot, A)$ in any meaningful way. However, we can say much more about the underlying systems $\{\succeq_A\}$. Moreover, this is, in our view, the conceptually interesting piece of the representation since it is what models the behavior of interest, viz. the DM’s preference formation process. Consequently, what we aim to identify here is the set of local menu preferences $\{\succeq_A\}$ for which there is an attached cardinal index $u(\cdot, \cdot)$ that represents $\succeq$. Identification of the object $\{\succeq_A\}$ also suffices for comparative statics analysis. In short, all of the ordinal data in the representation is identifiable. Moreover, since context-dependence is an ordinal condition this data suffices to link variation in context-dependence to variation in welfare, i.e. comparative statics.

The identification comes in three steps. First, we verify that, unless a menu preference is Arrowian in a sense defined below, it cannot admit a representation with an upwards monotonic utility index and, at the same time, another representation with a downwards monotonic and non-constant utility index. The second step of

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\(^{26}\)The straightforward proof is omitted.
identification shows that within the class of LTP (resp. LPF) representations \( \{ \succeq A \} \) there is a unique element \( \{ \succeq^* A \} \) such that any other LTP (LPF) representation is a coarsening of \( \{ \succeq^* A \} \). Finally, the third step shows that the set of local menu preference representations forms a sub-lattice of the lattice of systems of orders, \( \{ \{ \succeq A \} : \succeq A \in P(X) \} \) which is generated by (i.e. can be algorithmically reconstructed from) the minimal element.\(^{27}\)

**Definition 5.** Let \( \{ \succeq_A \} \), \( \{ \succeq'_A \} \) be two systems of menu-indexed orders. Say that the system \( \{ \succeq'_A \} \) coarsens the system \( \{ \succeq_A \} \) if, for every menu \( A \), we have that the local order \( \succeq'_A \) coarsens the local order \( \succeq_A \).

This defines a partial order on systems \( \{ \succeq_A \} \) which we will refer to as the **coarsening criterion**. We say that a menu preference \( \succeq \) is Arrowian if \( U(A) := \max_{x \in A} u(x) \) represents \( \succeq \), where \( u(\cdot) \) is any cardinal representation of \( \succeq \) on singleton menus. Let \( \Sigma_* \) denote the set of Arrowian preferences in \( P(X) \) that satisfy the Non-Emptiness axiom. Also let \( \Sigma_{LTP} \) (resp. \( \Sigma_{LPF} \)) denote the subset of preferences in \( P(X) \) that admit an LTP (resp. LPF) representation. Now consider preferences in this class that additionally admit a representation \( u(\cdot, \cdot) \) where (i) \( u(\cdot, \cdot) \) is upwards monotonic and (ii) \( \Phi_u(A) = C(A) \). Call this subclass \( \Sigma_{LTP}(f) \). Similarly define the subclass \( \Sigma_{LPF}(t) \) to be those preferences in \( \Sigma_{LPF} \) that additionally admit a representation \( u(\cdot, \cdot) \) where (i) \( u(\cdot, \cdot) \) is non-constant and downwards monotonic and (ii) \( \Phi_u(A) = C(A) \).

**Theorem 3** (Identification). Let \( \Gamma_{LPF}(\succeq), \Gamma_{LTP}(\succeq) \) denote (resp.) the sets of local menu preference representations of \( \succeq \).

1. \( \Sigma_{LTP}(f) = \Sigma_* \) (resp. \( \Sigma_{LPF}(t) = \Sigma_* \)).

2. The sets \( \Gamma_{LPF}(\succeq), \Gamma_{LTP}(\succeq) \) are lattices under the coarsening criterion and generated by explicitly constructible minimal elements \( \{ \succeq^*_A \} \).

The equality \( \Sigma_{LTP}(f) = \Sigma_* \) says that the monotonicity property is an “identifiable” property of preference for flexibility in the following sense. If a menu preference admits a local temptation preference representation and another representation with an upwards monotonic map, then the menu preference collapses to the value function of the ranking on singletons. A similar statement holds for menu preferences which admit a local preference for flexibility representation. In this case, non-constancy of the local utility implies the equality \( \Sigma_{LPF}(t) = \Sigma_* \).

The second part says that there is a unique, minimal representation such that every other is a coarsening of this representation. Moreover, the minimal element generates the lattice of representations. In other words, taking only the minimal

\(^{27}\)The relation on systems of orders that induces the lattice structure is the coarsening relation.
element as input, we construct an algorithm that recovers the lattice of representations: Every output of the algorithm is in the lattice and, conversely, every element of the lattice is recovered as an output of the algorithm. The reconstruction procedure is non-trivial and involves several steps. First, in the proofs of Theorems 1 and 2 we explicitly construct the most refined system \( \preceq A \). Second, we present the algorithm that inductively constructs a family of representations taking only the minimal element, \( \{ \succeq_A \} \), as input. Third, we formally prove that this algorithm recovers every representation (Proposition 4). The explicit description of the minimal element is important for two reasons. For the algorithm to be well-defined we need a construction of the minimal element that does not reference the other elements in the lattice. The other reason is that comparative statics are carried out on the minimal representation and require properties particular to the minimal representation. Without an explicit construction, there is no way to verify these properties.

By selecting the minimal representation for comparative statics we are making this the focal representation. The reason is that, in addition to generating the lattice, the minimal representation is also the most intuitive representation.\(^{28}\) For example, indifference classes of the local order are determined using only (i) the arg-max sets \( \Phi_u(A) \) and (ii) the inductive consistency condition. This allows an apples-to-apples comparison between two decision-makers: If one DM exhibits more preference for flexibility than another, measured by relative coarseness of their minimal systems \( \{ \succeq_A \} \), then we can interpret the extra coarseness as arising from a greater degree of context-dependence. Were we to select the minimal representation for one DM and a non-minimal element for another DM, then this interpretation is confounded by the fact that non-minimal elements involve additional coarsening that is not implied by either choices or the inductive consistency condition.\(^{29}\)

## 4 Comparative Statics

We present a pair of results. First, we characterize menu preferences that are represented by both our model and the Kreps (1979) model. This then leads to two comparative statics exercises. The first exercise is for the class of preferences that lie at the intersection of the LPF model and the Kreps model. The second is a more general comparative static for the full class of LPF preferences. While the second result is more general, the parameter on which the comparative static is formulated is the system of local orders, \( \{ \succeq_A \} \). This is a less tractable object than the cor-

\(^{28}\)Some properties are common to all representations in \( \Gamma_{LPF}(\succeq) \) (resp. \( \Gamma_{LTP}(\succeq) \)). For instance, any two representations, \( \{ \succeq_A \}, \{ \succeq_A' \} \), have the property that \( \{ x : x \succeq_A y, \forall y \in A \} = \{ x : x \succeq_A' y, \forall y \in A \} \), i.e., ex post choices are the same.

\(^{29}\)See the algorithm constructing the non-minimal representations for a more rigorous description of how this coarsening arises.
responding parameter for the first comparative static, which is the set of maximal fixed points of the critical points map \( C(\cdot) \). The maximal fixed points admit a concrete characterization and can be read off from the single order \( \succeq_X \) (they are the \( \succeq_X \)-indifference classes) as opposed to a family \( \{\succeq_A\} \).

To study the intersection of the LPF and Kreps models, we need some structural information. Recall that a subset \( B \subseteq X \) is an order interval in \( X \) (w.r.t the singleton ranking) if, writing \( X := \{x_1 \succeq x_2 \succeq \cdots \succeq x_n\} \), we have \( B = \{x_k, x_{k+1}, \ldots, x_l\} \) for some pair \((k,l)\) where \( 1 \leq k \leq l \leq n \). Let \( \Phi := \{A \in \mathcal{M} : C(A) = A\} \) denote the set of fixed points of the critical points map.

**Proposition 1 (Partition Structure).** Assume \( \succeq \in \mathcal{P}(X) \) satisfies \( A_1^*, A_2 \), and \( A_3 \). Let \( \{B_1, \ldots, B_k\} \) denote the maximal (w.r.t. set inclusion) elements of \( \Phi \), labeled so that \( B_i \succeq B_j \) when \( i < j \). Put \( \Phi_i := \{A \in \mathcal{M} : C(A) \subseteq B_i\} \). Then, the \( B_i \) are disjoint order intervals in \( X \) and \( A \succ B \) whenever \( A \in \Phi_i, B \in \Phi_j \) and \( i < j \).

The sets \( \Phi_i \) are relevant since they form a disjoint cover of \( \mathcal{M} \), i.e. (i) \( \Phi_i \cap \Phi_j = \emptyset \) and (ii) \( \bigcup_i \Phi_i = \mathcal{M} \). These facts follow from the proof of the Proposition. For the next step we recall the following well-known axiom. In contrast to Sen’s \( \beta \), this is a “contraction consistency” condition. Usually it is applied to an abstract choice correspondence, but here we are applying it to the critical points map \( C(\cdot) \).

Sen’s \( \alpha \): If \( A \subseteq B \), then \( C(B) \cap A \subseteq C(A) \).

Ergin (2003) has shown that this axiom can be swapped with Kreps’ Modularity \( (A2^*) \) to yield an equivalent characterization of Krepsian preference for flexibility. The appendix reproduces Ergin’s proof of this result. Let \( \Sigma_{LPF} \) denote the class of menu preferences which admit an LPF utility representation and let \( \Sigma_{KPF} \) denote those that are representable by a Kreps utility.

**Proposition 2 (Ergin (2003)).** Let \( \Sigma_{KPF^*} \) be the subset of \( \mathcal{P}(X) \) satisfying Monotonicity and Sen’s \( \alpha \). Then, \( \Sigma_{KPF^*} = \Sigma_{KPF} \).

Let \( \{B_i\} \) be a partition of \( X \) into disjoint order intervals, labeled so that \( B_i \succeq B_j \) if \( i < j \). Put \( \Sigma_i := \{A | A \subseteq B_i\} \) and define an element \( \succeq \in \mathcal{P}(X) \) as follows:

1. \( (\succeq)|_{\Sigma_i} \) is strictly monotonic. That is, \( B \succ A \), whenever \( A \nsubseteq B \) and \( A, B \in \Sigma_i \).

2. \( A \sim A \cap B_i^* \), where \( i^* = \min \{i : A \cap B_i \neq \emptyset\} \).

Denote the subclass of \( \mathcal{P}(X) \) satisfying (1) and (2) via \( \Sigma_{LSPF} \), i.e. “locally strict preference for flexibility”. The first property says the order is strictly monotonic on any nested pair of menus contained in the same partition cell \( B_i \). The second says that the indifference class of a menu is determined by the highest rank partition cell which intersects the menu.
Corollary 2. \( \Sigma_{\text{LPF}} \cap \Sigma_{\text{KPF}} = \Sigma_{\text{LSPF}} \).

The proof of this fact is immediate from Propositions 1 and 2, hence omitted. For a concrete model whose underlying menu preference lies in \( \Sigma_{\text{LSPF}} \) (aside from the Kreps utility) consider the utility class \( U(A) = |\Phi_u(A)| \cdot \max_{x \in A} u(x) \), i.e. there is no context-dependence, only a preference for freedom of choice. In particular, consider maps \( u : X \to \mathbb{R}_+ \) with \( u(x) > |X| \cdot u(y) \) whenever \( u(x) > u(y) \), and let \( U(A) := |\Phi_u(A)| \cdot \max_{x \in A} u(x) \) denote the associated menu utility. This generates a menu preference in \( \Sigma_{\text{LSPF}} \), with partition cells \( \{B_i\} \) equal to the \( u(\cdot) \)-indifference classes. In general, the partitions defining an LSPF preference coarsen the \( u(\cdot) \)-indifference classes.

Definition 6. Let \( \succeq_1, \succeq_2 \in \Sigma_{\text{LSPF}} \) agree on \( X \). Say that \( \succeq_1 \) has more preference for flexibility than \( \succeq_2 \), written \( (\succeq_1 \text{ MPF} \succeq_2) \), if whenever \( A \subset B \) and \( B \succ_2 A \), then \( B \succ_1 A \).

Proposition 3 (Comparative Statics, I). Let \( (\succeq_1) |_X = (\succeq_2) |_X \) and \( \succeq_1, \succeq_2 \in \Sigma_{\text{LSPF}} \), with (resp.) partitions \( \{B^1_i\}, \{B^2_i\} \). Then, \( (\succeq_1 \text{ MPF} \succeq_2) \) if and only if \( \{B^1_i\} \) is a coarsening of \( \{B^2_i\} \).

The assumption that \( \succeq_1 \) and \( \succeq_2 \) agree on the singleton ranking is required for the coarsening criterion to serve as a test of MPF. Consider \( \succeq_1, \succeq_2 \in \Sigma_{\text{LSPF}} \) with \( \{y\} \succ_1 \{z\} \succ_1 \{w\} \) and take

\[ \{B^1_i\} \equiv \{\{y\}, \{z\}, \{w\}\} \]

Now put \( \{w\} \succ_2 \{z\} \succ_2 \{y\} \) and let agent 2’s partition be given by

\[ B^2_1 = \{w\}, B^2_2 = \{y, z\}. \]

Notice that \( \{B^2_i\} \) is a coarsening of \( \{B^1_i\} \); however, it is neither the case that \( (\succeq_1 \text{ MPF} \succeq_2) \) nor that \( (\succeq_2 \text{ MPF} \succeq_1) \). To see this put \( A = \{y\}, B = \{y, z\} \) and note that \( B \succ_2 A \), but \( A \sim_1 B \). Similarly, put \( \hat{A} = \{w\}, \hat{B} = \{z, w\} \) and note that \( \hat{B} \succ_1 \hat{A} \) but \( \hat{A} \sim_2 \hat{B} \) so that \( \sim(\succeq_2 \text{ MPF} \succeq_1) \). Given a menu preference, \( \succeq_i \), we let \( \{\succeq^i_A\} \) denote the minimal element of the lattice \( \Gamma_{\text{LPF}}(\succeq_i) \).

Theorem 4 (Comparative Statics, II). Let \( (\succeq_1) |_X = (\succeq_2) |_X \) and \( \succeq_1, \succeq_2 \in \Sigma_{\text{LSPF}} \), with local menu preferences \( \{\succeq^1_A\}, \{\succeq^2_A\} \). Then, \( (\succeq_1 \text{ MPF} \succeq_2) \) if and only if \( \{\succeq^1_A\} \) coarsens \( \{\succeq^2_A\} \).

Let us examine the condition that \( \{\succeq^1_A\} \) coarsens \( \{\succeq^2_A\} \). Take a fixed pair \( (x, y) \) and imagine that both DM’s rank \( x \) strictly greater than \( y \) relative to some menu \( A \). Now take a larger menu \( B \) and say that DM 2 coarsens his ranking between \( x \) and \( y \), so that \( x \sim^B_2 y \). The result says that DM 1 also makes the switch from strict to weak. These weak preference reversals are exactly the gap between inductive consistency and deductive consistency. Hence, the result formalizes the idea that DM 1’s local preferences exhibit more inductive (i.e. context-dependent) reasoning than DM 2’s local preferences if and only if he exhibits greater desire for flexibility. In other words, context-dependence provides a source for preference for flexibility.
5 Conclusion

This paper has developed an axiomatic model of context-dependent preferences. Our idea is that a context-dependent decision-maker only formulates preferences over consumption options in the choice problem he is currently facing. Hence, as we vary choice problems we trace out a system of menu-indexed, or local, preferences, \( \preceq_A \). The main property of the orders \( \preceq_A \) that formalizes context-dependence is a condition called \textit{inductive consistency}. This condition allows preferences to be coarsened as we add choices. The intuition for this is that choices are based on reasons, hence as we add more options to the menu we implicitly also add reasons to support the choice of previously unselected options. The presence of these additional reasons induces ties in the local relations \( \preceq_A \), causing coarsening of the preference. While both choices and welfare are determined by the local menu preferences \( \preceq_A \), these preferences are unobservable to the modeler. The main exercise of the paper is to derive and identify this object by offering the DM choices between menus.

The representation results of the paper focus on preference for flexibility (Theorem 1) and preference for commitment (Theorem 2), and find axioms on the menu preference that characterize when it is generated by a system of local orders \( \preceq_A \) satisfying the inductive consistency property. We then show that, in both settings, the family of subjective local orders \( \preceq_A \) is identified (Theorem 3). There is usually more than one representation \( \preceq_A \), but the set of such representations forms a lattice and we explicitly reconstruct the lattice of representations from its minimal element (which is also explicitly constructed). Using the minimal representation, we carry out (Theorem 4) a comparative static that describes when one DM has a greater desire for flexibility than another in terms of variation in the local menu preferences.
6 Appendix

6.1 Proofs for Section 2

Proof of Theorem 1. Let \((u, \Phi_u)\) denote any LPF representation. Let us make a preliminary observation about these representations that will be useful in the forthcoming arguments: If we have an LPF representation \((u, \Phi_u)\), then for each menu \(A\), the function \(u(\cdot, A)\) is a coarsening of the singleton ranking (w.r.t. the underlying menu preference). While this will not be needed for the proof of necessity, it is used in the identification argument. Moreover, our construction in the sufficiency argument checks inductive consistency by first showing that the map we construct has this coarsening property. Finally, we note that the same property (with the same proof) holds for the LTP model. To verify the property, we use the fact (proven below) that \(C(A) = \Phi_u(A)\). That is, the set of critical elements in a menu are precisely the elements of the local arg maxima (w.r.t. the local order \(\succeq\)). Now consider any doubleton menu \(\{x, y\}\) where \(\{x\} \succeq \{y\}\). We claim that \(x \succeq_{\{x, y\}} y\). By inductive consistency, it would then follow that \(x \succeq_A y\) for any menu with \(x, y \in A\). Since \(\Phi_u(\{x, y\}) = C(\{x, y\})\), we have \(C(\{x, y\}) \neq 0\). If \(\{y\} = C(\{x, y\})\), then we obtain \(\{y\} \succeq \{x, y\} \succ \{x\} - contradiction. Hence, \(x \in C(\{x, y\})\) which implies, since \(C(\{x, y\}) = \Phi_u(\{x, y\})\), that \(x \succeq_{\{x, y\}} y\).

Necessity

We claim that we have the equality

\[ \Phi_u(A) = C(A), \forall A \in \mathcal{M}. \]

To see this, take \(x \in \Phi_u(A)\) and consider the utility of the menu \(A \setminus x\). By definition, we have \(U(A \setminus x) = |\Phi_u(A \setminus x)| \cdot u(y, A \setminus x)\), where \(y \in \Phi_u(A \setminus x)\). Consider separately the case where \(\Phi_u(A)\) is singleton and the case where it is non-singleton. In the case where it is singleton, by technical condition I(i) and the non-negativity of \(u(\cdot, \cdot)\), we have

\[ u(x, A) > |X| \cdot u(y, A \setminus x) \geq |\Phi_u(A \setminus x)| \cdot u(y, A \setminus x), \forall y \in A \setminus x. \]

It follows that \(U(A) > U(A \setminus x)\), showing that \(\Phi_u(A) \subseteq C(A)\). In the case where it is non-singleton find \(y(\neq x) \in \Phi_u(A) \cap (A \setminus x)\). Note that if \(z \in \Phi_u(A \setminus x)\), then \(z \succeq_A y\) so that \(z \succeq_A y - \text{by inductive consistency. It follows that } \Phi_u(A \setminus x) \subseteq \Phi_u(A)\). Hence,

\[ U(A) = |\Phi_u(A)| \cdot u(y, A) > |\Phi_u(A \setminus x)| \cdot u(z, A \setminus x) = U(A \setminus x) \]

where \(z \in \Phi_u(A \setminus x)\). The inequality follows from (i) \(u(z, A) \geq u(z, A \setminus x) > 0\) by upwards monotonicity (and \(\text{Im}(u) \subseteq \mathbf{R}_{++}\)), and (ii) the fact that, when \(\Phi_u(A)\) is non-singleton, \(\Phi_u(A \setminus x) \subseteq \Phi_u(A)\). Hence, in all cases, \(\Phi_u(A) \subseteq C(A)\). For the reverse inclusion, take \(x \in C(A)\) and towards contradiction say that \(x \notin \Phi_u(A)\). By technical condition I(ii), we then obtain \(u(y, A) = u(y, A \setminus x), \forall y \in \Phi_u(A)\). We claim
that we have $\Phi_u(A) \subseteq \Phi_u(A \setminus x)$. To see this, say that $y \in \Phi_u(A)$ (where $y \neq x$). If there is some $z \in A \setminus x$ such that $u(z, A \setminus x) > u(y, A \setminus x)$, then by upwards monotonicity we obtain $u(z, A) > u(y, A \setminus x) = u(y, A)$ – contradicting the hypothesis that $y \in \Phi_u(A)$. Hence, $\Phi_u(A) \subseteq \Phi_u(A \setminus x)$. Now take any $y \in \Phi_u(A)$ and notice that we have

$$U(A) = |\Phi_u(A)| \cdot u(y, A) \leq |\Phi_u(A \setminus x)| \cdot u(y, A \setminus x) = U(A \setminus x)$$

contradicting the hypothesis that $x \in C(A)$.

Given the equality $C(\cdot) = \Phi_u(\cdot)$, necessity of $A3$ is trivial. For necessity of $A1^*$, notice that – by definition – we have $A > A \setminus x$ for any $x \in C(A)$. If $x \not\in C(A)$, then by applying the preceding argument we have $\Phi_u(A) \subseteq \Phi_u(A \setminus x)$. Inductive consistency requires that $\succeq_{A}$ coarsens $\succeq_{A \setminus x}$, hence given that $\Phi_u(A) \subseteq \Phi_u(A \setminus x)$ we must in fact have equality, $\Phi_u(A) = \Phi_u(A \setminus x)$. It follows that $U(A) = U(A \setminus x)$, proving necessity of $A1^*$. For necessity of $A2$, we make critical use of the equality of $C(\cdot)$ with $\Phi_u(\cdot)$. Let $A \subseteq B$ and consider $y \in C(B) \cap A$. Then, $x \succeq_y y, \forall x \in C(A)$ (as $\Phi_u(A) = C(A)$). By inductive consistency, we obtain $u(x, B) \geq u(y, B)$. Since $u(y, B) \geq u(z, B), \forall z \in B$ this implies $x \in \Phi_u(B) = C(B)$, so that $C(A) \subseteq C(B)$.

**Sufficiency**

We break the proof into two main steps, (i) construction of the map $u(\cdot, \cdot)$ and (ii) verification that the constructed map satisfies the inductive consistency condition (the verification of the other conditions that define an LPF representation will be evident from the construction).

**Construction of $u(\cdot, \cdot)$**

Let $\Phi := \{A \in M : C(A) = A\}$ be the set of fixed points of the critical points map. Proceed in three (sub)steps.

**Step 1. Reduction to $\Phi$**

We claim that $A \sim C(A)$. Let $A \setminus C(A) = \{y_1, \ldots, y_k\}$ and put $A_i := A \setminus \{y_1, \ldots, y_i\}$. By $A2$, $C(A_i) \subseteq C(A), \forall i$. Iterative application of $A1^*$ then yields $A \sim A_1 \sim \cdots \sim A_k = C(A)$. Next, we check that $C(A) = C(C(A))$ via contradiction. Let $x \in C(A) \cap C(C(A))$ and note that $C(C(A)) \sim C(A) \sim A \sim A \setminus x$. Since $C(C(A)) \subseteq A \setminus x$, by $A1^*$ we obtain $A \setminus x \supseteq C(C(A))$. Thus, $C(C(A)) \sim C(A) \sim A \sim A \setminus x \supseteq C(C(A))$ – a contradiction.

**Step 2. Constructing the Representation on $\Phi$**

Put $K := |X|$ and select a cardinal representation $U(\cdot)$ of $\succeq$ with the property that for any $A, B \in \Phi$ with $B > A$ we have $U(B) > K \cdot U(A)$. Now put $u_{sp}(x, A) := \frac{U(A)}{|A|}, \forall A \in \Phi$. Notice that if we have $A' \subseteq A$ with $A', A \in \Phi$, then by definition of
$U(\cdot)$ (and $A1^*$) we have $u_\Phi(x,A) > u_\Phi(x,A')$ – so that $u_\Phi$ is upwards monotonic on $\Phi$. This defines an LPF representation of the preference on the set $\Phi$. Denote this representation as $u_\Phi(\cdot, \cdot)$.

**Step 3. Extension**

Now we extend the definition of $u_\Phi(\cdot, \cdot)$ to all menus. Put $u(x,A) := u_\Phi(x,C(A))$. Since $C(C(A)) = C(A)$, $u(x,A)$ is well-defined for all $x \in C(A)$. Put $\hat{U}(A) := \sum_{x \in C(A)} u(x,A)$ and notice that $\hat{U}(A) = U(C(A))$ so that $\hat{U}(\cdot)$ represents $\succeq$ by Step 1. We now check that the extended map is upwards monotonic. Put $A \subseteq B$. If $C(B) \cap A \neq \emptyset$, then $A2$ implies $C(A) \subseteq C(B)$. Hence, $u(x,B) := u_\Phi(x,C(B)) \geq u_\Phi(x,C(A)) =: u(x,A)$. Now extend $u(x,A)$ to $A \setminus C(A)$ as follows:

$$u(x,A) := \max\{u(x,A'): x \in C(A'), A' \subseteq A\}$$

and observe that $u(x,A)$ is upwards monotonic. This concludes the construction of the candidate LPF representation, $u(\cdot, \cdot)$.

**Verification of Inductive Consistency**

**Step 1.** Since the verification of inductive consistency is involved, we break into several sub-steps. In this step, we verify that $C(\cdot)$ is the arg max of the kernel $u(\cdot, \cdot)$ (this is also implied by the representation – as shown in the necessity argument). Since we have extended the definition of the map $u(\cdot, \cdot)$ to non-critical elements, we need to formally verify that we haven’t introduced non-critical elements into the arg max.

Note that we have $u(x,A) = u(y,A), \forall x, y \in C(A)$. Hence, we just need to check that $u(x,A) > u(y,A), \forall x \in C(A), y \in A \setminus C(A)$. Find $A' \subseteq A$ such that $A' = C(A')$ and $u(y,A) = u(y,A')$. Now consider two possible cases, (i) $A' \cap C(A) = \emptyset$ and (ii) $A' \cap C(A) \neq \emptyset$. In the latter case, by $A2$, we obtain $C(A') \subseteq C(A) \subseteq A$ – contradicting the hypothesis that $y \in A \setminus C(A)$. Hence, consider case (i). By deleting elements of $C(A)$ and applying $A3$ and then $A1^*$, we obtain: $A \succ A \setminus x \succeq A \setminus C(A)$, for any $x \in C(A)$. By $A1^*$ again, we have $A \setminus C(A) \succeq A'$ (as $C(A) \cap A' = \emptyset$), whence:

$$C(A) \sim A \succ A \setminus C(A) \succeq A' = C(A').$$

By selection of the cardinal representation $U(\cdot)$ it follows that $u(x,A) = u(x,C(A)) > u(y,C(A')) = u(y,A')$. This shows $C(A)$ is the arg max.

**Step 2.** We verify that $\succeq_A$, i.e. the order underlying $u(\cdot,A)$, coarsens the singleton ranking. For this part we invoke the partition structure of the preference, see Proposition 1. It suffices to show that $u(x) \geq u(y)$ (where $u(\cdot)$ represents $(\succeq)|_X$) implies $u(x,A) \geq u(y,A)$ (where $x,y \in A$). Note that if $x, y \in C(A)$, this is obvious as $u(x,A) = u(y,A)$. If $x \in C(A)$ and $y \in A \setminus C(A)$, then apply Step 1 to obtain that $u(x,A) > u(y,A)$. Moreover, the partition structure of the preference (Proposition 1) implies that $u(x) > u(y)$, so that $u(\cdot,A)$ is consistent with $u(\cdot)$ in this case. Next, consider $x, y \in A \setminus C(A)$. By symmetry, it is wlog to
put $u(x) \geq u(y)$. Find subsets $A_1, A_2 \subseteq A$ such that $x \in C(A_1), y \in C(A_2)$ and $u(x, A) = u(x, A_1), u(y, A) = u(y, A_2)$. Consider $A' := A_1 \cup A_2$. Since $C(A') \subseteq A'$ and $C(A') \neq \emptyset$ (by A3), we obtain that either $C(A') \cap A_1 \neq \emptyset$ or $C(A') \cap A_2 \neq \emptyset$. Thus, by A2 either $x \in C(A')$ or $y \in C(A')$. If the former holds, then

$$u(x, A') \geq u(y, A') \geq u(y, A_2) = u(y, A)$$

The middle inequality follows by upwards monotonicity, which also implies $u(x, A) \geq u(x, A')$ — so that $u(x, A) \geq u(y, A)$. Next, allege $y \in C(A'), x \notin C(A')$ and consider the menu $\{x, y\} \subseteq A'$. By A2, we know that $C(\{x, y\}) \subseteq C(A')$ so that $x \notin C(\{x, y\})$. Hence, applying A3, we get $\{y\} \succeq \{x, y\} \succ \{x\}$ — which contradicts the hypothesis that $u(x) \geq u(y)$. Hence, $u(\cdot, A)$ coarsens $u(\cdot)$.

**Step 3a.** We now use Step 2 to check the inductive consistency property. Fix $A \subseteq B$ with $x, y \in A$. We will show that $u(x, A) \geq u(y, A)$ implies $u(x, B) \geq u(y, B)$. Note that if $u(x, A) > u(y, A)$, then since $u(\cdot, A)$ is a coarsening of $u(\cdot)$ we must have $u(x) > u(y)$, implying that $u(x, B) \geq u(y, B)$ as $u(\cdot, B)$ is a coarsening of $u(\cdot)$ as well. Thus, it remains to check the case where $u(x, A) = u(y, A)$. If $u(x) = u(y)$, then since $u(\cdot, B)$ coarsens $u(\cdot)$ we obtain $u(x, B) = u(y, B)$. By symmetry, it is wlog to put $u(x) > u(y)$. Since $u(x, A) = u(y, A)$ find $A_1 \subseteq A$ with $y \in C(A_1)$ and $u(y, A_1) = u(y, A)$. Consider $A^* = A_1 \cup \{x\}$ and note that, by A3, we must have either (i) $\{x\} = C(A^*)$ or (ii) $C(A^*) \cap A_1 \neq \emptyset$. The former case cannot occur as $u(x, A) = u(y, A)$. Hence, $C(A^*) \cap A_1 \neq \emptyset$, which implies (by A2) $y \in C(A^*)$. Since $u(x) > u(y)$ this then implies $x \in C(A^*)$. The menu $A^*$ and the fact that $x, y \in C(A^*)$ will be invoked in the next step.

**Step 3b.** Now return to the menu $B$, where $A \subseteq B$. Recall that we want to show $u(x, B) = u(y, B)$. Let $B_1 \subseteq B$ be such that $x \in C(B_1)$ and $u(x, B_1) = u(x, B)$. Consider the menu $\hat{B} := B_1 \cup A^*$. Note that (by A3) either (i) $C(\hat{B}) \cap A^* \neq \emptyset$ or (ii) $C(\hat{B}) \cap B_1 \neq \emptyset$. In case (i), we have (by A2) $C(A^*) \subseteq C(\hat{B})$, so that $x, y \in C(\hat{B})$. Hence, $u(x, \hat{B}) = u(y, \hat{B})$. Since $u(x, B) = u(x, \hat{B})$ and $u(y, B) \geq u(y, \hat{B})$ (by upwards monotonicity) we obtain $u(y, B) \geq u(x, B)$. Since $u(\cdot, B)$ coarsens $u(\cdot)$ and $u(x) > u(y)$, we have $u(x, B) \geq u(y, B)$. Hence, $u(x, B) = u(y, B)$. In case (ii), we have $C(B_1) \subseteq C(\hat{B})$ (by A2). Hence, $x \in C(\hat{B})$. As $x \in A^*$, this implies $C(\hat{B}) \cap A^* \neq \emptyset$, so that (by A2) $C(A^*) \subseteq C(\hat{B})$. Hence, $y \in C(\hat{B})$. This implies $u(x, B) = u(y, B)$ and, in turn, $u(x, B) = u(y, B)$.

**Proof of Theorem 2.** The proof of Necessity, in this case, is more or less trivial on account of the (much) stronger technical condition(s). OTOH, on account of the added richness of the preference (i.e. the coexistence of preference for flexibility over consumption plans and preference for commitment over consumption choices) the sufficiency argument is more involved than in the LPF case.
Necessity
Let $\succeq$ be the menu preference generated by the LTP representation $(u, \Phi_u)$. We first verify the equality $C(A) = \Phi_u(A)$. To see this, let $x \in \Phi_u(A)$ and note that by technical condition II(i) we have
\[
|\Phi_u(A \setminus x)| \cdot u(y, A \setminus x) < |\Phi_u(A)| \cdot u(x, A), \forall y \in A \setminus x.
\]
Hence, $x \in C(A)$. Conversely, let $x \in C(A)$. Towards contradiction, say that $x \notin \Phi_u(A)$. Then, by technical condition II(ii), we have
\[
|\Phi_u(A)| \cdot u(x, A) \leq |\Phi_u(A \setminus y)| \cdot u(x, A \setminus y), \forall y \in A \setminus x.
\]
Hence, $U(A \setminus x) \geq U(A) - $ contradiction. It follows that $C(A) = \Phi_u(A)$. Given this equality, we now verify necessity of the axioms. Note that $u(\cdot, A)$ is non-constant, so there is some $y \in A \setminus \Phi_u(A)$ (for $A$ non-singleton). Since $C(A) = \Phi_u(A)$, this implies $C(A) \neq A$ – so that $A_1$ follows. Non-emptiness $(A_3)$ similarly follows as $\Phi_u(A)$ is always non-empty. For $A_2$ let $A \subseteq B$ and put $y \in C(B) \cap A$. Take any $x \in C(A) \setminus \Phi_u(A)$. By definition, $x \succeq_A y$. By inductive consistency, $x \succeq_B y$. Since $C(B) = \Phi_u(B)$, $y \succeq_B z, \forall z \in B$. Hence, $x \succeq_B z, \forall z \in B$, so that $x \in \Phi_u(B) = C(B)$. It follows that $C(A) \subseteq C(B)$.

Sufficiency
Let $A_1, A_2, \ldots$ be any enumeration of $\mathcal{M}$ and identify a utility $U(\cdot)$ on $\mathcal{M}$ with a vector in the Euclidean space $\prod_A R_i$ (here $R_i$ denotes the component of the vector which gives the utility of menu $A_i$). We will find an LTP representation of $\succeq$ in a neighborhood of $\vec{1} \in \prod_A R_i$. Let
\[
\kappa := \frac{1}{|X|} - \frac{1}{|X| + 1}
\]
and choose a cardinal representation $U(\cdot)$ of $\succeq$ in $B_\epsilon(\vec{1})$ where $\epsilon < \kappa/2$. Put $U(A) = 1 + \epsilon_A$ and define,
\[
u(x, A) = \frac{1 + \epsilon_A}{|C(A)|}, \forall x \in C(A).
\]
This defines the map on elements of $C(A)$. To extend to other elements of the menu in a manner that satisfies inductive consistency and downwards monotonicity constitutes the heart of the proof. As in the proof of Theorem 1 we break the forthcoming argument into two main steps. The first is the construction of the candidate LTP map, $u(\cdot, \cdot)$. The second step then verifies that the constructed map satisfies the defining properties of an LTP representation. Since the critical points map $C(\cdot)$ has no non-singleton fixed points, we cannot proceed by first constructing $u(\cdot, \cdot)$ on fixed point menus – as in the LPF case. A different construction is required.
Construction of $u(\cdot, \cdot)$

We construct an equivalence relation on singletons $x \in A$ which yields a partition of $A$ into order intervals. The construction generalizes the one given in Proposition 1 to accommodate the presence of temptation. Since we are assuming $A1$ and this precludes ties in the singleton ranking, we will henceforth assume the singleton ranking is strict. Make the following definition:

**Definition 7.** Put $xR_A y$ if there is a menu $A' \subseteq A$ with $x, y \in C(A')$. If $xR_A y$, we say that elements of the pair $(x, y)$ are *not separated.*

$R_A$ is a well-defined binary relation on the set $A$ that is clearly symmetric and reflexive. We claim that $R_A$ is transitive. Take $(x, y, z)$ with $xR_A y, yR_A z$. Let $A_1, A_2$ be sub-menus such that $\{x, y\} \subseteq C(A_1), \{y, z\} \subseteq C(A_2)$. Consider the menu $A' = A_1 \cup A_2$ and note that $C(A') \neq \emptyset$ by $A3$. Hence, either $C(A') \cap A_1$ or $C(A') \cap A_2 \neq \emptyset$. If the former holds, then $\{x, y\} \subseteq C(A_1) \subseteq C(A')$ by $A2$. Hence, $y \in C(A') \cap A_2 \neq \emptyset$ and we obtain $\{y, z\} \subseteq C(A_2) \subseteq C(A')$. Hence, $\{x, z\} \subseteq C(A')$, implying that $xR_A z$. Similarly, argue if $C(A') \cap A_2$, then $C(A_2) \subseteq C(A')$ which implies $\{x, z\} \subseteq C(A')$, so that $xR_A z$. Hence, the binary relation $R_A$ is transitive. It follows that the $R_A$-equivalence classes form a partition of $X$.

Let $\{B_1, \ldots, B_n\}$ be an enumeration of the $R_A$-equivalence classes. We claim that the $B_i$ are order intervals. Let $x, y \in B_i$ with $\{x\} \succ \{z\} \succ \{y\}$. We check that $z \in B_i$. Let $A'$ be such that $\{x, y\} \subseteq C(A')$ and consider the menu $A^* = A' \cup \{z\}$. Since $\{x\} \succ \{z\}$ and $x \in A'$ it follows that $C(A^*) \cap A' \neq \emptyset$, implying that $\{x, y\} \subseteq C(A^*)$. Since $C(A^*)$ is an order interval in $A^*$ (see Step 1 of proof of Theorem 3) it follows that $z \in C(A')$, so that $xR_A z$ and $z \in B_i$. Hence, the collection $\{B_1, \ldots, B_n\}$ partitions $A$ into order intervals.

Let $U(A) = 1 + \epsilon_A$ denote the cardinal utility selected earlier. Define

$$u(x, A) = \begin{cases} \frac{1+\epsilon_A}{|C(A)|}, & \text{if } x \in C(A) \\ u'(x, A), & \text{if } x \in (A\setminus C(A)) \cap B_i. \end{cases}$$

where $u'(x, A)$ is a function we define shortly. This is a subtle aspect of the construction, so we first give some intuition for the formula we eventual write down for $u'(x, A)$. The reader who wants to skip the intuition can pass directly to the definition of the map $u(\cdot, \cdot)$ (see here) without loss of continuity. Consider the not-separated relations, $R_A$. Note that in the $R_A$-induced partition of $A$ into order

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30 This is the part of the argument that shares some overlap with the construction in Tyson (2008). Please see the attached supplement for a more precise description of this overlap.
intervals, the top cell of the partition is all of $C(A)$.\(^{31}\)

Consider the quantities, $u(x, A) := \min\{\frac{U(A')}{|C(A)|} : A' \subseteq A, x \in C(A')\}$. Downwards monotonicity requires that, for each $y \in B_i$, the quantity $\overline{u}(x, A)$ yields an upper bound on the value $u'(x, A)$. For each $x \in \cup_{i>1} B_i$, let $\overline{u}(x, A) := \max\{\frac{U(A')}{|C(A)|} : A' \supseteq A, x \in C(A')\}$. Downwards monotonicity requires that this is a lower bound on the value of $u'(x, A)$. Hence, we need to be able to choose $u'(x, A)$ such that

$$u(x, A) \leq u'(x, A) \leq \overline{u}(x, A).$$

To see that this can be done, let $A' \subseteq A$ be such that $C(A') = B_i$, where $x \in B_i$ and $B_i$ is the $R_A$-equivalence class of $x$. Temporarily ignoring the $1 + \epsilon_A$ terms in the numerator we obtain that

$$u(x, A) \leq \frac{1}{|B_i|}. $$

Moreover, for any $x \in B_i$ with $A' \subseteq A$ and $x \in C(A')$ we have $C(A') \subseteq B_i$. Hence,

$$\overline{u}(x, A) = \frac{1}{|B_i|}. $$

Note that this equality holds for all $x \in B_i$. Next we find an upper bound for $u(x, A)$. Consider $\hat{A} \supseteq A$ with $x \in C(\hat{A})$. Let $A'$ denote a subset of $A$ with $B_i = C(A')$. Note that if $x \in C(\hat{A})$, then $C(\hat{A}) \cap A' \neq \emptyset$. Hence, $C(A') \subseteq C(\hat{A})$. Also, since $C(A')$ is an order interval in $A'$ we obtain that (putting $x \in B_i$)

$$\cup_{j \leq i} B_j \subseteq C(\hat{A}). $$

Hence, recalling that $B_1 = C(A)$, we obtain that

$$u(x, A) \leq \frac{1}{|C(A)| + \sum_{j=2}^{i} |B_j|}. $$

It follows that

$$\left[\frac{1}{|C(A)| + \sum_{j=2}^{i} |B_j|}, \frac{1}{|B_i|}\right] \subseteq [u(x, A), \overline{u}(x, A)]$$

\(^{31}\)To see this, let $\{B_1, \ldots, B_k\}$ denote the partition of $A$ into its $R_A$-equivalence classes labeled so that $\text{sup}(A) \in B_1$. Since $C(A)$ is an order interval in $A$ with $\text{sup}(A) \in C(A)$, it follows that $C(A) \subseteq B_1$. Now if $(A \setminus C(A)) \cap B_1 \neq \emptyset$, then let $z$ be in the intersection and find a menu $A' \subseteq A$ with $\{x, z\} \subseteq C(A')$, where $x \in C(A)$. By $A2$ this implies that $C(A') \subseteq C(A)$ – contradiction. Hence, $B_1 = C(A)$. 36
for $x \in B_i$. Also note that if we choose the value of $u'(x, A)$ to be the lower endpoint of this sub-interval, then we also satisfy $u'(x, A) < \frac{U(A)}{|C(A)|}$ (by choice of $\kappa$ and since $U(\cdot) \in B_i(\bar{U})$). This now leads us to our definition of the extended map $u(\cdot, \cdot)$.

$$u(x, A) = \begin{cases} \frac{U(A)}{|C(A)|}, & \text{if } x \in C(A) \\ \frac{1 + \epsilon'_A}{|C(A)| + \sum_{j=2} |B_j|}, & \text{if } x \in (A \setminus C(A)) \cap B_i. \end{cases}$$

where $\epsilon'_A$ is a to-be-determined positive constant that is at most $\kappa/2$. This is our candidate LTP representation $u(\cdot, \cdot)$. By selection of $\kappa$ (see the case (i) argument below), we have the equality: $U(A) = |\Phi_u(A)| \cdot u(x, A)$. Moreover, since $\Phi_u(A) = C(A)$, $u(\cdot, \cdot)$ satisfies technical condition II. We now check inductive consistency and downwards monotonicity.

**Verification of Inductive Consistency**

**Step 1.** As in the LPF proof, we first check that $u(\cdot, A)$ coarsens $u(\cdot)$. Take $u(x) > u(y)$ with $x, y \in A$. If $x, y \in C(A)$, then $u(x, A) = u(y, A)$ by construction. Similarly, if $x \in C(A)$ and $y \in A \setminus C(A)$, we have $u(x, A) > u(y, A)$. Note that $x \in A \setminus C(A), y \in C(A)$ cannot hold as $C(A)$ is an order-interval in $A$ and $\sup(A) \in C(A)$. Hence, consider $x, y \in A \setminus C(A)$ and let $\{B_i\}$ denote the $R_A$-equivalence classes. Note that if $x, y \in B_i$, then from definition of $u(x, A)$ we have $u(x, A) = u(y, A)$. If $x, y$ are in different partition cells, then we must have $x \in B_i, y \in B_j, j > i$ as $u(x) > u(y)$. By choice of $\kappa$ and since $\epsilon'_A \leq \kappa/2$, this implies $u(x, A) > u(y, A)$ (see case (i) argument below for details).

**Step 2.** We now verify inductive consistency. Take $A \subseteq B$ and let $x, y \in A$. If $u(x, A) > u(y, A)$, then since $u(\cdot, A)$ coarsens $u(\cdot)$ we must have $u(x) > u(y)$. Hence, as $u(\cdot, B)$ coarsens $u(\cdot)$ we must have $u(x, B) \geq u(y, B)$. Thus, consider $u(x, A) = u(y, A)$. We check that $u(x, B) = u(y, B)$ as well. By symmetry, take $u(x) > u(y)$. Note that $u(x, A) = u(y, A)$ if and only if $xR_A y$. Since $A \subseteq B$, we have $xR_B y$ implying that $u(x, B) = u(y, B)$.

**Verification of Downwards Monotonicity.** Fix $A \subseteq B$ and consider four cases:

i. $x \in C(A), x \notin C(B)$.

ii. $x \notin C(A), x \in C(B)$.

iii. $x \in C(A), x \in C(B)$.

iv. $x \notin C(A), x \notin C(B)$.
Case (i). Let $B_i$ denote the $R_B$-equivalence class of $x$ and note that, since $A \subseteq B$, $C(A) \subseteq B_i$. Put $u(x, A) = \frac{1 + \epsilon_A}{|C(A)|}$ and notice that

$$u(x, B) = \frac{1 + \epsilon'_B}{|C(B)|} \leq \frac{1 + \epsilon'_B}{1 + |B_i|} \leq \frac{1 + \epsilon_A}{|C(A)|}.$$ 

To verify (*) note that $|\epsilon_A|, \epsilon'_B \leq \kappa/2$. Hence, it suffices to check that:

$$\frac{1 + \kappa/2}{1 + |B_i|} < \frac{1 - \kappa/2}{|B_i|}.$$ 

Equivalently, we check that

$$\frac{\kappa/2}{1 + |B_i|} + \frac{\kappa/2}{|B_i|} < \frac{1}{|B_i|} - \frac{1}{1 + |B_i|}.$$ 

Since $|B_i| \geq 1$, the LHS is bounded above (strictly) by $\kappa$ and by definition

$$\kappa = \frac{1}{|X|} - \frac{1}{|X| + 1}$$

so that the inequality (**) follows. Hence, $u(x, B) \leq u(x, A)$ in this case.

Case (ii). In the second case, note that by A2 we have $C(A) \subseteq C(B)$. Moreover, if $x \in C(B)$, then $\bigcup_{j \leq i} B_j \subseteq C(B)$ where $\{B_j\}_{j=2}^i$ denotes the $R_A$-equivalence classes of elements of $A \setminus C(A)$ consisting of elements whose singleton rank is higher than $x$ along with the $R_A$-class of $x$. It follows that

$$|C(A)| + \sum_{j=2}^i |B_j| \leq |C(B)|. \quad (1)$$

Since $u(x, B) = \frac{1 + \epsilon_B}{|C(B)|}$, $u(x, A) = \frac{1 + \epsilon'_A}{|C(A)| + \sum |B_j|}$, $|\epsilon_B|, \epsilon'_A \leq \kappa/2$, by the same argument as in case (i) we obtain $u(x, B) \leq u(x, A)$ if (1) holds with strict inequality. Now consider the equality case of (1). For this, we need to carefully define the $\epsilon$-factor that is in the numerator of $u(x, A) := \frac{1 + \epsilon'_A}{|C(A)| + \sum |B_j|}$. This can be chosen, due to our choice of $\kappa$, to be a small amount (weakly) greater than $\epsilon_B$, where $u(x, B) = \frac{1 + \epsilon_B}{|C(B)|}$. Proceed as follows: For $x \in B_i \subseteq A \setminus C(A)$ consider $\epsilon'_A := \frac{\kappa}{2} \cdot 1_{\Sigma_A}$, where

$$\Sigma_A = \{ B : A \subseteq B, C(B) = C(A) \cup \bigcup_{j=2}^i B_j \}.$$ 

Put

$$u(x, A) := \frac{1 + \epsilon'_A}{|C(A)| + \sum_{j=2}^i |B_j|}.$$
The indicator variable equals 0 if and only if $\Sigma_A$ is empty, in which case the equality case of (1) doesn’t occur.

**Case (iii).** In this case, the inequality $u(x, A) \geq u(x, B)$ is obvious from choice of the neighborhood $B_i(\bar{1})$ if $C(A) \not\subseteq C(B)$. Consider the possibility that $C(A) = C(B)$. Put $B \setminus A = \{y_1, \ldots, y_k\}$ and set $A_i = B \setminus \{y_1, \ldots, y_i\}$. Note that $A_1 \supseteq B$ and, by $A_2$, $C(A_1) \subseteq C(B) \subseteq A$. Moreover, $C(A_1) \not= \emptyset$ by $A_3$. Iteratively apply the same argument to each $A_i$ and obtain $B \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k = A$. Hence, we obtain: $U(A) = 1 + \epsilon_A \geq 1 + \epsilon_B = U(B)$. Since $u(x, A) = \frac{1 + \epsilon_A}{|C(A)|}$, $u(x, B) = \frac{1 + \epsilon_B}{|C(B)|}$ and $C(A) = C(B)$, we obtain $u(x, A) \geq u(x, B)$.

**Case (iv).** Applying the same argument as in case (i), by choice of $\kappa$ and since $\epsilon_A' \leq \kappa/2$, we obtain:

$$u(x, A) := \frac{1 + \epsilon_A'}{\sum_{j=1}^i |B_j|} = \frac{1 + \epsilon_A'}{|\{z \in A : u(z, A) \geq u(x, A)\}|}$$

for any menu $A$, where $x \in B_i$ and the $\{B_j\}$ denote the $\mathcal{R}_A$-equivalence classes. Observe that inductive consistency of $u(\cdot, \cdot)$ implies

$$\{z \in A : u(z, A) \geq u(x, A)\} \subseteq \{z \in A : u(z, B) \geq u(x, B)\}.$$

Hence, if $\epsilon_A' \geq \epsilon_B'$, then it follows that $u(x, A) \geq u(x, B)$. However, it may be the case that $\epsilon_A' \not= 0, \epsilon_B' = \kappa/2$. This happens precisely when $\Sigma_B \not= \emptyset, \Sigma_A = \emptyset$. Let $B_i$ denote the $\mathcal{R}_A$-class of $x$ and $\overline{B}_j$ denote its $\mathcal{R}_B$-class. Note that if $j < i$ (so that $B_j$ is a higher ranked $\mathcal{R}_A$-class), then $B_j \subseteq \cup_{j \leq i'} \overline{B}_j$ – again, by inductive consistency. Hence, if $D \in \Sigma_B$, then

$$\cup_{j \leq i} B_j \subseteq \cup_{j \leq i'} \overline{B}_j = C(D).$$

If $\Sigma_A = \emptyset$, then – by definition – the containment is strict, implying that $u(x, A) > u(x, B)$ – by choice of $\kappa$ and since $\epsilon_A' \leq \kappa/2$. It follows that, in all cases, $u(x, A) \geq u(x, B)$, i.e. $u(\cdot, \cdot)$ is downwards monotonic.

### 6.2 Proofs for Section 3

**Proof of Theorem 3.** First, we verify identification of the monotonicity property. Second, we prove that the representations constructed in the proofs of Theorems and 2 are the uniquely minimal representations (under the coarsening criterion). Third, we prove the lattice structure of the set of representations. The second point is superfluous from the point of view of just getting a proof of the result. Once we have the lattice structure, since there are finitely many elements in the lattice of representations, it is obvious that there is a unique minimum and maximum. Our
Identification of Upwards/Downwards Monotonicity

We first verify the equality $\Sigma_{LTP}(f) = \Sigma_*$. If $\succeq \in \Sigma_{LTP}(f)$, let $u(\cdot, \cdot)$ be a non-constant representation of $\succeq$ with the property that $\Phi_u(A) = C(A)$. For any menu $A$ we have:

$$|C(A)| \cdot u(x, A) = U(A) = U(C(A)) = |C(C(A))| \cdot u(x, C(A)), \forall x \in C(A).$$

Since $C(C(A)) = C(A)$ we must have $u(x, A) = u(x, C(A)), \forall x \in C(A)$. It follows that $u(\cdot, C(A))$ is constant on $C(A)$, which implies, since $u(\cdot, \cdot)$ is non-constant, $C(A)$ must be a singleton for all menus $A$. Hence, $\succeq \in \Sigma_*$. Next, verify that $\Sigma_{LTP}(f) = \Sigma_*$. Let $\succeq \in \mathcal{P}(X)$ satisfy $A_1 \prec A_3$ and let $A = \{x_1, \ldots, x_k\}$ be a top-down (w.r.t. $\succeq |_X$) enumeration of the elements of the menu $A$. Proceed in 3 steps.

Step 1. We check that $x_1 \in C(A)$ and that $C(A)$ is an order interval in $A$. This requires only $A_2$ and $A_3$. Let $\{x_1\} \succeq \{x_j\}$ and assume $x_j \in C(A)$. Consider the menu $A' := \{x_i, x_j\}$. We claim that $x_i \in C(A')$. Towards contradiction, if $x_i \notin C(A')$, then $\{x_i\} = A' \setminus x_i \succeq A'$. On the other hand, by $A_3$, $C(A') \neq \emptyset$ so that $C(A') = \{x_j\}$. Thus, $A' \succ A' \setminus x_j = \{x_i\}$. Put together this yields: $\{x_j\} = A' \setminus x_i \succeq A' \setminus x_j = \{x_i\}$ – which is a contradiction. Thus, $x_i \in C(A')$. Since $x_j \in C(A) \cap A'$, $A_2$ implies $C(A') \subseteq C(A)$ so that $x_i \in C(A)$.

Step 2. We check that $C(A) \succeq A$. By Step 1 and $A_1$, the $(\succeq |_X)$-minimal element of a menu is not critical. This shows $\{x_1\} \succeq \{x_1, x_2\} \succeq \cdots \succeq \{x_1, x_2, \ldots, x_k\}$. By Step 1 and $A_1$, $C(A) = \{x_1, \ldots, x_j\}$ for some $j < k$. It follows that $C(A) \succeq A$.

Step 3. We check that $A \sim C(A) = \sup(A), \forall A \in \mathcal{M}$. Assume that $\succeq \in \Sigma_{LTP}(f)$, and let $u(\cdot, \cdot)$ be an upwards monotonic function such that $U(A) = |\Phi_u(A)| \cdot u(x, A)$ represents $\succeq$ and $\Phi_u(A) = C(A)$. Proceed by induction on the cardinality of the menu. Consider $A = \{x_1, x_2\}$, where $\{x_1\} \succ \{x_2\}$ ($A_1$ and Step 1 implies a strict singleton ranking). Note that $C(A) = \{x_1\}$ by Step 2 and $A_1$. Since $U(A) = u(x_1, A)$, by upwards monotonicity of $u(\cdot, \cdot)$ we have $A \succeq \{x_1\}$. On the other hand, $\{x_1\} = C(A) \succeq A$, so that $C(A) \sim A$. Assume the claim holds for all menus of cardinality less than or equal to $k$ and let $A$ have cardinality $k + 1$. Put $A = \{x_1, \ldots, x_{k+1}\}$, where $\{x_1\} \succ \{x_2\} \succ \cdots \succ \{x_{k+1}\}$. If, towards contradiction, $x_j \in C(A)$ for some $j \neq 1$, then $A \succ A \setminus x_j$. By the induction hypothesis,
implies

\[ \{x_1\} = C(C(A)) \sim C(A) \supseteq A \succ A \setminus x_j \sim C(A \setminus x_j) = \{x_1\} \]

which yields our contradiction. Thus, \(C(A) = \{x_1\}\). Since \(u(\cdot, \cdot)\) represents \(\succeq\), we obtain \(U(A) = u(x_1, A) \geq u(x_1, C(A)) = U(C(A))\). Since \(C(A) \succeq A\) this then implies \(A \sim C(A) = \{x_1\}\).

**Identification of \(\succeq_A\): LPF case.**

Let \(\{\succeq_A\}\) be the system constructed in the proof of Theorem 1. That is, we take \(u(x, A)\) to be defined on fixed point menus such that the technical condition is satisfied and extend to all menus via the formula:

\[
u(x, A) := \max\{u(x, A') : x \in C(A'), A' \subseteq A\}.
\]

(2)

The underlying ordinal system of orders that \(u(\cdot, \cdot)\) represents is denoted \(\{\succeq_A\}\). Now let \(\{\succeq_A\}\) be any other system that represents the same menu preference \(\succeq\). We claim that \(\{\succeq_A\}\) coarsens \(\{\succeq_A^*\}\). To see this, fix a menu \(A\) and recall the relations \(\mathcal{R}_A\) constructed in the proof of Theorem 2. If \(x, y \in A\) and \(x \mathcal{R}_A y\), then we must have \(x \sim_A y\) by the inductive consistency condition, for any local menu preferences \(\{\succeq_A\}\) that represent \(\succeq\). We now show that

\[
x \sim_A^* y \iff x \mathcal{R}_A y.
\]

(3)

The direction \(x \mathcal{R}_A y \Rightarrow x \sim_A^* y\) follows from inductive consistency. For the converse, let \(A_x, A_y \subseteq A\) be menus such that the maximum \(u(x, \cdot)\) is attained on the menu \(A_x\) (and similarly for \(A_y\)). Wlog say that \(\{x\} \succeq \{y\}\). Consider the menu \(A' := A_x \cup A_y\). Notice that \(C(A') \neq \emptyset\) (by A3), so that \(C(A') \cap A_x \neq \emptyset\) or \(C(A') \cap A_y \neq \emptyset\). In the former case, we have (by A2) \(x \in C(A')\), so that \(x \sim_A^* z, \forall z \in A'\). OTOH, taking \(u(\cdot, \cdot)\) to be the cardinal representation of \(\succeq_A^*\) defined in (2), we know that \(u(x, A_x) = u(x, A) \geq u(x, A') \geq u(x, A_y)\) – the latter two inequalities following from upwards monotonicity. Hence, \(u(x, A) = u(x, A')\). Similarly, we find: \(u(y, A_y) = u(y, A) \geq u(y, A') \geq u(y, A_y)\), so that \(u(y, A) = u(y, A')\). Since \(u(x, A) = u(y, A)\) it follows that \(u(x, A') = u(y, A')\), which implies that \(y \succeq_A^* z, \forall z \in A'\). Hence, \(y \in C(A')\) and we obtain \(x \mathcal{R}_A y\). Now consider the case where \(C(A') \cap A_y \neq \emptyset\). By A2 we have \(y \in C(A')\). Since \(\{x\} \succeq \{y\}\) this implies \(x \in C(A')\), so that \(x \mathcal{R}_A y\). It follows that \(x \sim_A^* y \iff x \mathcal{R}_A y\). Hence, if \(x \sim_A^* y\), then \(x \sim_A y\) for any local system \(\{\succeq_A\}\) that represents \(\succeq\). Now consider the case where \(x \succ_A y\). We check that \(x \succeq_A y\). Note that \(x \succ_A y\) implies that \(\{x\} \succ \{y\}\) – as local menu preferences coarsen the singleton ranking. It follows that if \(\{\succeq_A\}\) is a system of local menu preferences that represents \(\succeq\), then we must also have \(x \succeq_A y\). This proves that \(\succeq_A^*\) is a coarsening of \(\succeq_A^*\).
Identification of \{\succeq_A\}: LTP case.
Let \{\succeq_A^*\} denote the local menu preference constructed in the proof of Theorem 2.
As in the preceding proof, the key observation is
\[ x \sim_A^* y \iff x \mathcal{R} y \]
where \mathcal{R}_A is the “not-separated” relation constructed in the proof of Theorem 2.
This equivalence was non-obvious from the definition of the function \(u(\cdot,\cdot)\) in the LPF setting, but it is obvious from the definition in the LTP setting. Now let \{\succeq_A\} be any other system of local menu preferences that represents \(\succeq\). We claim that \{\succeq_A\} is a coarsening of \{\succeq_A^*\}. For any menu \(A'\), if \(x, y \in C(A')\), then we must have \(x \sim_{A'} y\) for any LTP system, \{\succeq_A\}, that represents \(\succeq\). It follows, from inductive consistency, that if \(x \mathcal{R}_A y\) then \(x \sim_A y\). Since \(x \sim_A^* y \iff x \mathcal{R}_A y\) we obtain:
\[ x \sim_A^* y \implies x \sim_A y \]
Next, we verify that:\(x \sim_A^* y \implies x \sim_A y\). Since \(\sim_A^*\) coarsens the singleton ranking, if \(x \sim_A^* y\), then \(\{x\} > \{y\}\). Since all local menu preferences coarsen the singleton ranking, we obtain \(x \succeq_A y\).\(^{32}\) Hence, \(\succeq_A\) coarsens \(\succeq_A^*\).

Lattice Structure. Let \(\Gamma_{\text{LPF}}(\succeq), \Gamma_{\text{LTP}}(\succeq)\) denote the sets of LPF (resp. LTP) representations, i.e. \(\{\succeq_A\} \in \Gamma_{\text{LPF}}(\succeq)\) if there is a cardinal utility index \(u(\cdot,\cdot)\) which represents \(\{\succeq_A\}\) and generates the menu preference. The argument here is nearly identical for the sets \(\Gamma_{\text{LPF}}(\succeq), \Gamma_{\text{LTP}}(\succeq)\), hence we suppress the LPF/LTP subscript in what follows. Also, for this proof alone, let \(\mathcal{L} := \{\succeq_A\}\) denote a generic element of \(\Gamma(\succeq)\). Define a partial order \(\succeq\) on \(\Gamma(\succeq)\) (the coarsening criterion) as follows:
Say that \(\mathcal{L}_1 \succeq \mathcal{L}_2\) if and only if the system \(\mathcal{L}_1\) coarsens \(\mathcal{L}_2\) menu-by-menu. Define \(\mathcal{L}_1 \wedge \mathcal{L}_2\) as follows. Let \([y, x] := \{z \in X : \{z\} \succeq \{y\}\}\). For each menu \(A\), put \(x \sim_A^* y \iff \exists z_0 = x, z_1, \ldots, z_k = y \text{ s.t. } z_i \sim_A^* z_{i+1}\), for some \(i = 1, 2\) (with strict preference else). In words, \(x\) is indifferent to \(y\) if and only if there is a chain of singletons between \(x\) and \(y\) with a chain of indifferences \(x \sim_A^* z_1, \ldots, z_{k-1} \sim_A^* y\), where the preferences along the chain of indifferences can alternate between agents \(i = 1, 2\) (i.e. they need not be drawn from the same agent \(i\)). Note that \(\succeq_A^*\) coarsens the singleton ranking, hence this formula yields a well-defined coarsening of \(\succeq_A\).
Note that \(x \succeq_A^* y\) is a common coarsening of \(\succeq_A^1, \succeq_A^2\). Doing this menu-by-menu we clearly obtain a system \(\mathcal{L} := \{\succeq_A^*\}\) of inductively consistent orders such that \(\mathcal{L} \succeq \mathcal{L}_1, \mathcal{L}_2\). Moreover, by definition, any common coarsening of \(\mathcal{L}_1, \mathcal{L}_2\) clearly must coarsen \(\mathcal{L}\). It follows that \(\mathcal{L} = \mathcal{L}_1 \wedge \mathcal{L}_2\). To see that this is also a representation of \(\succeq\) note that the sets \(\Phi_u(A)\) are identical for each \(\succeq_A, \succeq_A^*\) comprising \(\mathcal{L}_1, \mathcal{L}_2\). Hence, \(\mathcal{L}\) is also a local menu preferences representation of \(\succeq\) \((n.b: \text{there is a subtle point involving the cardinal index } u(\cdot,\cdot)\text{ that supports this system as a representation, relegated to the remark following the proof})\).

\(^{32}\)This follows from the equality \(C(A) = \Phi_u(A)\). See the remarks preceding proof of Theorem 1.
Finally, define the element \( L_1 \lor L_2 \) as follows. Put \( x \sim_A^y \leftrightarrow [\forall z_0 = x, z_1, \ldots, z_k = y, z_i \sim_A z_{i+1}, \text{for some } i = 1, 2] \) (with strict preference else). In other words, \( x \) is indifferent to \( y \) if and only if there is no chain going from \( x \) down to \( y \) with a strict preference (for some agent) along some point in the chain. Note that the system \( L := \{ \succeq_A' \} \) is well-defined and inductively consistent. Moreover, it is a refinement of \( L_1, L_2 \) so that \( L \preceq L_1, L_2 \). By definition, it also follows that any other refinement of both \( L_1, L_2 \) must also refine \( L \). Hence, \( L = L_1 \lor L_2 \). Moreover, by the same reasoning as before, this is also a representation of \( \succeq \). It follows that \( \Gamma(\succeq) \) is a lattice. Completeness is obvious and we have also (constructively) verified existence of a unique minimal element. For the maximal element note that the set \( \Gamma(\succeq) \) is finite, so we simply take \( L := \bigwedge_{L_i \in \Gamma(\succeq)} L_i \) to obtain the unique, coarsest representation. This concludes the proof of parts (1) and (2) of the Theorem. The construction of the algorithm and the proof that it recovers the lattice of representations, part (3) of the Theorem, is given following the remark below. 

Remark: Given two representations \( L_1, L_2 \) we have shown that \( L_1 \land L_2 \) and \( L_1 \lor L_2 \) yields a system of inductively-consistent orders whose top indifference classes, \( \Phi_u(A) \), equal those of \( L_1, L_2 \). We need to show we can attach a cardinal index to the meet and join that is resp. upwards/downwards monotonic. This follows as a by-product of the algorithmic reconstruction of the lattice given below. There we start with the minimal element of the lattice, along with the cardinal index \( u(\cdot, \cdot) \) constructed in the proofs of Theorems 1 and 2, and recover all representations in the lattice. This construction can then be mimicked to find the cardinal index that supports \( L_1 \land L_2 \) (resp. \( L_1 \lor L_2 \)) as a representation.

6.2.1 Algorithm

We now present an algorithm to recover the lattice of representations given the minimal representation. We first introduce some notation required to define the algorithm. Also note: the algorithm applies nearly verbatim for both the LPF vs. LTP cases. There are only two places in the construction where a change is required to address the LTP case and these are footnoted.

1. For each \( n \) let \( \mathcal{M}(n) \) denote the set of menus of cardinality \( n \) and for each \( A \in \mathcal{M}(n), x \in A \) let \( A(x) \) denote the admissible set (defined below).

2. Fix a cardinal representation of the minimal element, denoted \( u_0(\cdot, \cdot) \), and let \( u_{n-1}(\cdot, \cdot) \) denote the map obtained after extending the to menus in \( \mathcal{M}(n) \).

The only piece of notation that takes the underlying menu preference as input (beyond the initial representation \( u_0 \)) is the definition of the sets \( A(x) \). We define these now. To explain the condition we write down let us ask the following question: taking the coarsening \( u_{n-1} \) from step \( n-1 \) of the algorithm as
given, which elements \( y \) can be coarsened with \( x \) at step \( n \) of the algorithm, i.e. \( u_{n-1}(x, A) > u_{n-1}(y, A) \) and \( u_n(x, A) = u_n(y, A)? \) The set of \( y \) that satisfy this property is determined by the menu preference, with constraints defined as follows. Let \( \overline{A}(x) := \cap_{B: A \subseteq B, x \in C(B)} C(B). \)

\[
y \in A(x) \text{ iff } [x \in \overline{A}(x) \Rightarrow y \in \overline{A}(x)]
\]

Note that the premise, \( x \in \overline{A}(x) \), is not vacuous since \( \overline{A}(x) \) may be empty. Let us explain this restriction. Note that all representations have the same set of ex post choices, i.e. the sets \( \Phi_n(A) \) agree across all representations. Since \( C(A) = \Phi_n(A) \) (for the constructed representation \( u_0(\cdot, \cdot) \)), we obtain that if \( x \in C(B) \) for some \( B \supseteq A \) and \( y \notin C(B) \), then we cannot have \( u_n(x, A) = u_n(y, A) \) for any putative coarsened representation – as inductive consistency would then require \( y \in C(B) \), which is a contradiction. Since any superset \( B \) has this property, the admissible set consists of the intersection of all such \( C(B) \). Note that, since all representations \( u_n \) coarsen the minimal representation, the following additional condition must hold: if \( u_{n-1}(y, A) \geq u_{n-1}(z, A) \) and \( z \in A(x) \), then \( y \in A(x) \). Hence, at stage \( n \) of the algorithm, for each \( A(x) \) there is minimal (w.r.t. \( u_{n-1}(\cdot, \cdot) \)) element, \( z(x) \), such that

\[
u_n(x, A) = u_n(y, A) = u_n(z(x), A), \forall y \in A(x)
\]

and \( u_{n-1}(y, A) \geq u_{n-1}(z(x), A) \). This discussion motivates the following algorithm.

**Description of Algorithm**

**Step 0:** (Initialization) Construct the minimal element of the lattice of representations, \( u_0(\cdot, \cdot) \).

**Step n:** (Coarsening on \( \mathcal{M}(n+1) \)) Let \( u_{n-1}(\cdot, \cdot) \) denote the representation obtained at the conclusion of Step \( n-1 \). Extend as follows.

1. For each \( A \in \mathcal{M}(n+1) \) and \( x \in A \) select any \( z(x) \in A(x) \).

2. Extend to \( \mathcal{M}(n+1) \) as follows. Put \( \preceq_{n-1} \) equal to the order on \( A \in \mathcal{M}(n+1) \) induced by \( u_{n-1}(\cdot, \cdot) \). For \( A \in \mathcal{M}(n) \):

\[
u_n(x, A) = \max_{x': x \in A(x'), x \geq_{n-1} z_n(x')} u_{n-1}(x', A).
\]

3. Extend to all menus as follows. Put \( x \rightarrow_A y \) if there is a chain \( x_0 = x, x_1, \ldots, x_k = y \) with \( u_n(x_i, A_i) = u_n(x_{i+1}, A_i) \) and \( u_{n-1}(x_i, A_i) \geq u_{n-1}(x_{i+1}, A_i) \), \( A_i \in \mathcal{M}(n+1) \). Put:

\[
u_n(y, A) := \max_{x \rightarrow_A y} u_{n-1}(x, A).
\]

\[^{33}\text{In the LTP case, we replace the max with min.}\]

\[^{34}\text{In the LTP case, we replace the max with a min and the condition } u_{n-1}(x, A_i) \geq u_{n-1}(x_{i+1}, A_i) \text{ is replaced with } u_{n-1}(x, A_i) \leq u_{n-1}(x_{i+1}, A_i).\]
Step N: End. (N = |X|)

Let \( \mathcal{A}(u_0) \) denote the set of coarsenings of \( u_0 \) produced via this algorithm and let \( \Theta(\mathcal{A}(u_0)) \) denote the set of underlying ordinal systems \( \{\succeq_A\} \). We verify that \( \Theta(\mathcal{A}(u_0)) = \Gamma(\succeq) \). This equality compresses two statements. First, the algorithm yields a representation of \( \succeq \) (in either the LPF/LTP case). This is what the containment \( \Theta(\mathcal{A}(u_0)) \subseteq \Gamma(\succeq) \) says. The second statement is that every representation in the lattice can be recovered via this algorithm, this is the containment \( \Gamma(\succeq) \subseteq \Theta(\mathcal{A}(u_0)) \).

**Proposition 4 (Characterization).** \( \Gamma(\succeq) = \Theta(\mathcal{A}(u_0)) \).

**Proof.** \( \Gamma(\succeq) \supseteq \Theta(\mathcal{A}(u_0)) \): Let \( u(\cdot, \cdot) \in \mathcal{A}(u_0) \). We first check that \( \Phi_u(A) = \Phi_{u_0}(A) \). Note that \( \Phi_{u_0}(A) = C(A) \). Hence, if \( x \in C(A) \), then by the admissibility restriction we must have \( z(x) \in C(A) \). It follows that \( u(x, A) = u_0(x, A) \), which shows that \( \Phi_{u_0}(A) = C(A) \subseteq \Phi_u(A) \). For the reverse, put \( x \in \Phi_u(A) \). Note that \( u(x, A) = \max_{x \in \Phi_{u_0}(A)} u_0(x, A) \). Hence, since \( \Phi_{u_0}(A) \subseteq \Phi_u(A) \), we obtain that \( x \in \Phi_{u_0}(A) \). It follows that we have \( U(A) := |\Phi_u(A)| \cdot u(x, A) = |\Phi_{u_0}(A)| \cdot u_0(x, A) \) for any \( x \in C(A) \) since \( \Phi_{u_0}(A) = \Phi_u(A) \) and \( u_0(x, A) = u(x, A) \) for any \( x \in C(A) \). Moreover, from the algorithm we know for any \( x \notin C(A) \) we have \( u(x, A) \leq \max_{x \notin C(A)} u_0(x, A) \). Since \( u_0(z, A) > |X| \cdot u_0(y, A) \) when \( \{z\} = \Phi_{u_0}(A) \) we obtain, \( u(z, A) = u_0(z, A) > |X| \cdot \max_{y \neq z} u_0(y', A) \) for \( y \neq z \). Finally, note that \( C(A) = C(A \setminus y) \). Hence, \( u_0(x, A \setminus y) = u(x, A \setminus y) \), \( \forall y \notin \Phi_{u_0}(A) = \Phi_u(A) \). It follows that the growth conditions (in the LPF case) are satisfied. Since \( \Phi_u(A) = C(A) \) for any \( u \in \mathcal{A}(u_0) \) the growth conditions in the LTP case are also satisfied.

**Verification of Inductive Consistency.** To check inductive consistency, we verify that \( u_n \) is inductively consistent assuming that \( u_{n-1} \) is inductively consistent. To show this, first check that if \( u_{n-1}(x, A) = u_{n-1}(y, A) \), then \( u_n(x, A) = u_n(y, A) \). Note that, if \( A \in \mathcal{M}(n+1) \), then \( u_n(x, A) = \max_{x': x \geq_{n-1} z_{n}(x')} \). Since

\[
\{x' : x \geq_{n-1} z_{n}(x')\} = \{x' : y \geq_{n-1} z_{n}(x')\}
\]

as \( x \sim_{n-1} y \), we obtain \( u_n(x, A) = u_n(y, A) \). If \( A \in \mathcal{M}(m) \) where \( m < n + 1 \), then \( u_n(\cdot, A) = u_{n-1}(\cdot, A) \). If \( A \in \mathcal{M}(m) \), where \( m > n + 1 \) then, by definition of \( u_n \) and inductive consistency of \( u_{n-1} \), we obtain \( u_n(x, A) = u_n(y, A) \). Fix \( A \subseteq B \). We check that \( u_n(x, A) \geq u_n(y, A) \) if \( u_n(x, B) \geq u_n(y, B) \). Note that if \( u_{n-1}(x, A) = u_{n-1}(y, A) \), then \( u_{n-1}(x, B) = u_{n-1}(y, B) \), by inductive consistency of \( u_{n-1} \). Applying the preceding argument we obtain that \( u_n(x, A) = u_n(y, A) \) and \( u_n(x, B) = u_n(y, B) \). Hence, consider \( u_{n-1}(x, A) > u_{n-1}(y, A) \). If \( u_{n-1}(x, B) = u_{n-1}(y, B) \), then we again obtain \( u_n(x, B) = u_n(y, B) \). Hence, we reduce to \( u_{n-1}(x, A) > u_{n-1}(y, A) \) and \( u_{n-1}(x, B) > u_{n-1}(y, B) \). By definition
of $u_n$ and inductive consistency of $u_{n-1}$, it follows that $u_n(x, A) \geq u_n(y, A)$ and $u_n(x, B) \geq u_n(y, B)$.

**Verification of Upwards/Downwards Monotonicity.** This is the only step of the argument where we need to invoke the distinction in the algorithm between the LPF vs. LTP case. First consider the LPF case. If $A \subseteq B$ we check that $u(x, A) \leq u(x, B)$. Assuming $u_{n-1}$ is upwards monotonic, we check that $u_n$ must be as well. Consider three cases as before, (i) $A = u_{n-1}(x, A)$ the argument where we need to invoke the distinction in the algorithm between the LPF and LTP cases. Many arguments are similar to the upwards monotonic case, so we will be deliberately terse. As before, consider the three cases (i) $A \in M, m < n$, (ii) $A \in M(n)$, and (iii) $A \in M(m), m > n$. In the first case we have $u_n(x, A) = u_{n-1}(x, A)$. Moreover, by construction, $u_n(x, B) = \max_{z: z \rightarrow A} u_{n-1}(z, B) \geq u_{n-1}(x, B)$. Since $u_{n-1}(x, B) \geq u_{n-1}(x, A) = u_n(x, A)$, it follows that $u_n(x, B) \geq u_n(x, A)$. When $B \in M(n)$ we have, by definition, $u_n(x, B) \geq u_{n-1}(x, B)$ so that the same argument applies. In case (ii), we cannot necessarily appeal to an equality $u_n(x, A) = u_{n-1}(x, A)$. If this holds, then the argument for case (i) applies. However, if we have $u_n(x, A) > u_{n-1}(x, A)$ then find $x' \in A$ such that $u_n(x', A) = u_n(x, A)$ and $x'$ is $\geq_{n-1}$- maximal. Then, we obtain $u_n(x, B) = \max_{z: z \rightarrow B} u_{n-1}(z, B) \geq u_{n-1}(x', B) \geq u_{n-1}(x', A) = u_n(x, A)$, where the latter inequality invokes upwards monotonicity of $u_{n-1}$. Finally, in case (iii) note that $\{z : z \rightarrow B \} \supseteq \{z : z \rightarrow A \}$. Hence, let $z_A, z_B$ be the (resp.) maximizers of $u_{n-1}(\cdot, B), u_{n-1}(\cdot, A)$ on these two sets. We then obtain

$$u_n(x, B) = u_{n-1}(z_B, B) \geq u_{n-1}(z_A, B) \geq u_{n-1}(z_A, A)$$

where the first inequality follows from $z_A \in \{z : z \rightarrow B \}$ and the second from the hypothesis that $u_{n-1}$ is upwards monotonic. Since $u_{n-1}(z_A, A) = u_n(x, A)$ it follows that $u_n(x, B) \geq u_n(x, A)$. This proves upwards monotonicity of $u_n$ assuming upwards monotonicity of $u_{n-1}$.

We now assume $u_{n-1}$ is downwards monotonic and verify that $u_n$ is downwards monotonic. Many arguments are similar to the upwards monotonic case, so we will be deliberately terse. As before, consider the three cases (i) $A \in M(m), m < n$, (ii) $A \in M(n)$, and (iii) $A \in M(m), m > n$. In the first case, we have $u_{n-1}(x, A) = u_n(x, A)$. Since $u_n(x, B) \leq u_{n-1}(x, B)$ we have $u_n(x, A) = u_{n-1}(x, A) \geq u_{n-1}(x, B) \geq u_n(x, B)$, where the former inequality follows from the hypothesis that $u_{n-1}$ is downwards monotonic. In case (ii), if we have $u_n(x, A) = u_{n-1}(x, A)$, then the preceding argument applies verbatim. If, on the other hand, we have $u_n(x, A) < u_{n-1}(x, A)$, then find $\geq_{n-1}$-minimal $x' \in A$ with $u_n(x, A) = u_{n-1}(x', A)$ and $u_n(x', A) = u_{n-1}(x', A)$. The preceding argument implies $u_n(x', A) \geq u_n(x', B)$. Hence,

$$u_n(x, B) := \min_{x: z \rightarrow B} u_{n-1}(x, B) \leq u_{n-1}(x', B) \leq u_{n-1}(x', A) = u_n(x', A)$$

where the latter inequality invokes downwards monotonicity of $u_{n-1}$. Since $u_n(x', A) = u_n(x, A)$, it follows that $u_n(x, A) \geq u_n(x, B)$. In case (iii), we find $z_A, z_B$ which are minimizers of $u_{n-1}(\cdot, A)$ (resp. $u_{n-1}(\cdot, B)$) on the (resp.) sets $\{z : z \rightarrow A \}, \{z : z \rightarrow B \}$.
\[ z \rightarrow B \ x \}. \] Since \( \{ z : z \rightarrow_B x \} \supseteq \{ z : z \rightarrow_A x \} \), it follows that
\[ u_n(x, B) = u_{n-1}(z_B, B) \leq u_{n-1}(z_A, B) \leq u_{n-1}(z_A, A) = u_n(x, A) \]
where the latter inequality invokes downwards monotonicity of \( u_{n-1} \).

\[ \Gamma(\succeq) \subseteq \Theta(A(u_0)): \] Starting with a local menu preferences representation \( \{ \succeq_A \} \) we want to recover this system as a terminal element of the algorithm. The method of proof is as follows.\(^{35}\) Given a system \( \{ \succeq_A \} \in \Gamma(\succeq) \) we use induction to find an element \( A(u_0) \). Let \( \{ \succeq_A' \} \) be the system represented by \( u_0 \). Find a first \( n \) for which \( \{ \succeq_A \} \) does not agree with \( \{ \succeq_A' \} \) on \( M(n+1) \). We show that there are choices of thresholds \( z(x) \in A(x) \) for each \( A \in M(n+1), x \in A \) such that \( u_n(\cdot, \ A) \) agrees with \( \succeq_A \) restricted to \( M(m), \forall m \leq n + 1 \). Let \( \{ \succeq_A^n \} \) denote the orders underlying \( u_n \). We then show that there are choices of thresholds on the level \( M(n+1) \) such that \( \{ \succeq_A^{n+1} \} \) agrees with \( \{ \succeq_A \} \) on \( \cup_{m=1}^{n+1} M_m \). Inductively proceeding, we recover the system \( \{ \succeq_A \} \) as a terminal element of the algorithm. Take a minimal \( n_0 \) where \( \{ \succeq_A \} \) disagrees with \( \{ \succeq_A' \} \) on some menus in \( M(n_0 + 1) \). Since all local menu preferences \( \{ \succeq_A \} \) coarsens \( \{ \succeq_A' \} \), for each \( A \in M(n_0 + 1) \) and \( x \in A \) find the \( u_0 \)-minimal \( z_x \) such that \( x \sim_A z_x \) (if there are ties in the \( u_0 \) ranking, label elements in the indifference class and choose the element with the maximal label). Since \( \{ \succeq_A \} \) represents \( \succeq \) and \( \Phi_u(A) = \Phi_{u_0}(A) = C(A) \) we must have \( z_x \in A(x) \), i.e. the \( z_x \) are admissible. Put
\[ u'_{n_0}(x, A) := \max_{x' : x \succeq_0 z_x} u_0(x', A) \]
and, again, replace “max” with “min” when considering the LTP case. We claim that \( u'_{n_0}(\cdot, A) \) represents \( \succeq_A \). To see this, put \( y := z_x \) and note that \( z_y = y \) (by definition). Hence, we obtain \( u'_{n_0}(y, A) = u_0(z_x, A) \). Next, we note that \( x \succeq_A y \iff z_x \succeq_A z_y \) (again, by definition). It follows that if \( x \succ_A y \), then \( z_x \succ_A z_y \). Moreover, since \( \succeq_A \) coarsens \( \succeq_0 \) we must have: \( x \succeq_0 z_x \succ_0 y \succeq_0 z_y \), whenever \( x \succ_0 y \). Hence, if \( x \succ_A y \) we must have
\[ \{ x' : x \succeq_0 z_x \} \supseteq \{ x' : y \succeq_0 z_y \} \]
which implies that \( u'_{n_0}(x, A) \geq u'_{n_0}(y, A) \). If \( x \sim_A y \), then we claim that \( u'_{n_0}(x, A) = u'_{n_0}(y, A) \). By symmetry, take \( x \succeq_0 y \). Then, \( x \succeq_0 y \succeq_0 z_x = z_y \). Note that \( z_x = z_y \) for any \( x' \) in the \( u_0 \)-order interval \( [z_x, z_y] \). It follows that \( \{ x' : x \succeq_0 z_x \} = \{ x' : y \succeq_0 z_y \} \), implying \( u'_{n_0}(x, A) = u'_{n_0}(y, A) \).

For the reverse, note that \( u'_{n_0}(x, A) \geq u'_{n_0}(y, A) \) if and only if \( \{ x' : x \succeq_0 z_x \} \supseteq \{ x' : y \succeq_0 z_y \} \). To prove this, note that right-to-left is obvious from definition of the function \( u'_{n_0}(\cdot, A) \). Let us check the left-to-right implication. Assume \( u_{n_0}(x, A) \geq u_{n_0}(y, A) \) and let \( g(x), q(y) \) denote the (resp.) maximizers on the sets
\(^{35}\)Unlike the proof of the reverse inclusion, in this case there is no substantive point in the argument where the distinction between LPF vs. LTP representation is relevant.

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\{x^\prime : x \succeq_0 z_{x^\prime}\}, \{x^\prime : y \succeq_0 z_{x^\prime}\}. \text{ Notice that } q(x) \succeq_0 q(y) \Rightarrow q(x) \succeq_A q(y), \text{ as } \succeq_A \text{ coarsens } \succeq_0. \text{ Hence, } z_{q(x)} \succeq_A z_{q(y)}. \text{ By the argument presented above (using the fact that } z_{q(x)} = z_{q(y)}, z_{q(y)} = z_{q(y)}). \text{ This implies that } z_{q(x)} \succeq_0 z_{q(y)}. \text{ It follows that } \{x^\prime : x \succeq_0 z_{x^\prime}\} \supseteq \{x^\prime : y \succeq_0 z_{x^\prime}\}. \text{ Since } y \in \{x^\prime : y \succeq_0 z_{x^\prime}\} \text{ it follows that } x \succeq_0 z_y. \text{ Since } \succeq_A \text{ coarsens } \succeq_0 \text{ this implies } x \succeq_A z_y \text{ which, as } z_y \sim_A y, \text{ implies } x \succeq_A y.

Doing this for each menu } A \in \mathcal{M}(n_0+1) \text{ we obtain a coarsening, } u_{n_0}, \text{ of } u_0 \text{ that (i) corresponds to step } n_0 \text{ of the algorithm (in preceding steps we pick } z_x = x, \text{ so there is no coarsening of } u_0) \text{ and (ii) agrees with } \succeq_A, \forall A \in \bigcup_{n_0+1}^{m+1} \mathcal{M}(m). \text{ Now iterate this argument, replacing } u_0 \text{ with } u_{n_0}. \text{ Let } n_1 \text{ be the first level } \mathcal{M}(n_1 + 1), \text{ where } n_1 > n_0, \text{ for which there is disagreement between } \succeq_A \text{ and } u_{n_0}(\cdot, A) \text{ for some } A \in \mathcal{M}(n_1 + 1). \text{ Inductively extend using the same argument as above to a coarsening } u_{n_1} \text{ which (i) corresponds to level } n_1 \text{ of the algorithm (i.e. choose } x = z_x \text{ for menus on levels between } n_0 \text{ and } n_1 \text{) and (ii) agrees with } \succeq_A \text{ on } \bigcup_{m=1}^{n_1+1} \mathcal{M}(m). \text{ The terminal element of the sequence } u_{n_0}, u_{n_1}, \ldots \text{ produces a coarsening of } \{\succeq_A\} \text{ (the minimal element of the lattice) which is a cardinal representation of the given } \{\succeq_A\} \in \Gamma(\succeq).

6.3 Proofs for Section 4

Proof of Proposition 1.

Step 1. First, we check that the maximal fixed points \{B_i\} form a partition of \(X\). Note that \(\Phi \neq \emptyset\) so that maximal elements exist. We claim that the maximal fixed points, \(B_i\), are disjoint. Via contradiction, let \(B_1, B_2\) be maximal elements with non-empty intersection. Put \(B^* = B_1 \cup B_2\) and consider \(C(B^*)\). Since \(C(B^*) \subseteq B_1 \cup B_2\) we must have either \(C(B^*) \cap B_1 \neq \emptyset\) or \(C(B^*) \cap B_2 \neq \emptyset\) by A3. Say the former occurs. Then, \(C(B_1) \subseteq C(B^*)\) by A2. On the other hand, since \(B_1 \cap B_2 \neq \emptyset\) and \(B_i = C(B_i)\) we obtain \(C(B^*) \cap B_i \neq \emptyset\). A2 then yields \(C(B_j) \subseteq C(B^*)\). Thus, \(C(B^*) = C(B_1) \cup C(B_2) = B_1 \cup B_2 = B^*\). This contradicts the maximality of both \(B_i\) and \(B_j\). To show that \(\forall i, B_i = X\) note that any singleton menu \(\{x\}\) is in \(\Phi\). Thus, \(\{x\}\) is contained in some maximal \(B_i\).

Step 2. We first check the ordering on the sets \(\Phi_i\), then use this to verify the \(B_i\)'s are order intervals in \(X\). The preceding argument shows that \(C(B_1 \cup B_2) = B_i \cup B_j\). Label such that \(B_1 \succeq B_2 \cdots \succeq B_k\). By Claim 1, this implies \(C(B_1 \cup B_j) = B_i\) whenever \(i < j\). Now consider any \(A \subseteq B_i\) such that \(C(A) = A\). Since \(C(B_1 \cup B_j) = B_i\), by A2 we obtain \(C(A \cup B_j) \subseteq C(B_i \cup B_j) = B_i\). Since \(C(A \cup B_j) \subseteq A \cup B_j\) and \((A \cup B_j) \cap B_i = A = C(A)\), it follows that \(C(A \cup B_j) \subseteq C(A)\). On the other hand, since \(C(A \cup B_j) \cap A \neq \emptyset\), A2 implies \(C(A) \subseteq C(A \cup B_j)\). Thus, \(A = C(A) = C(A \cup B_j)\). It follows that, for any \(x \in A\), we have \(A \cup B_j \supseteq (A \setminus x) \cup B_j \supseteq B_j\) (by A1*), so that \(A \succ B_j\). If \(B \in \Phi_j\), then \(B_j \succeq C(B)\) (by A1*) and \(C(B) \sim B\), so that \(A \succ B\) – verifying the ordering on the sets \(\Phi_i\). We now use this property to check \(B_1\) is an order interval, a similar argument applies to
other \(B_i\). Towards contradiction, if \(B_1\) is not an order interval, then there is some pair \((x, y)\) such that \(x \in B_i, y \notin B_j\) (for some \(j > 1\)) where \(\{y\} \supseteq \{x\}\). However, applying the preceding argument with \(A = \{x\}\) we then obtain \(\{x\} \succ B_j \supseteq \{y\}\) (the latter relation by \(A1^*\)), which yields a contradiction.

**Claim 1.** Assume \(\succeq \in \mathcal{P}(X)\) satisfies \(A1^*, A2,\) and \(A3\) and let \(\Phi\) be the set of fixed points of the map \(C(\cdot)\) with maximal elements \(\{B_1, \ldots, B_k\}\). If \(B_i \succeq B_j\), then \(C(B_i \cup B_j) = B_i\).

**Proof.** Step 1 (which uses \(A2\) and \(A3\)) shows \(C(B_i \cup B_j) = B_i\) or \(C(B_i \cup B_j) = B_j\). Proceed via contradiction and say that \(C(B_i \cup B_j) = B_j\). Then, \(B_i \cup B_j \setminus y \succeq B_i \cup B_j\), for any \(y \in B_i\). By \(A2\), \(C(B_i \cup B_j \setminus y) \subseteq C(B_i \cup B_j) = B_j\). Thus, \(B_i \cup B_j \setminus \{y, z\} \succeq B_i \cup B_j \setminus B_j\), for any \(z \in B_i \setminus y\). Inductively, we obtain \(B_j \succeq B_i \cup B_j\). On the other hand, if \(C(B_j \cup B_i) = B_i\), then for any \(x \in B_j, B_i \cup B_j \succ B_i \cup B_j \setminus x \succeq B_i\), the latter relation following from \(A1^*\). Thus, \(B_j \succeq B_i \cup B_j \succ B_i\) which is a contradiction since the labeling was chosen such that \(B_i \succeq B_j\).

**Proof of Proposition 2.**

**Step 1.** We first show that \(\Sigma_{KPF^*} \subseteq \Sigma_{KPF}\). Let \(A \subset B\) with \(A \sim B\). It suffices to show that for any \(x \not\in B\) we have \(A \cup \{x\} \sim B \cup \{x\}\). Via contradiction, assume there is \(x\) for which this is false. That is, let \(B \cup \{x\} \succ A \cup \{x\}\) for some \(x \notin B\). Let \(D^*\) be a minimal (w.r.t set inclusion) subset of \(B \cup \{x\}\) containing \(A\) such that \(D^* \succ A \cup \{x\}\). Write \(D^* = \bigcup D' \cup \{x\}\). By \(A1^*\) and minimality of \(D^*\), \(D' \subseteq C(D^*)\). On the other hand, since \(D' \not\subseteq A\), let \(z \in D' \setminus A\) and consider \(A' = A \cup \{z\}\). Note that \(A' \subseteq D^*\). By Sen's \(\alpha\), \(z \in C(A')\). Thus, on the one hand: \(A' \succ A\). On the other hand, \(A \sim B \succeq A'\), the latter by \(A1^*\). Put together we obtain, \(A \sim B \succeq A' \succ A\), which is a contradiction.

**Step 2.** We check the reverse containment \(\Sigma_{KPF} \subseteq \Sigma_{KPF^*}\). Via contradiction again, let \(B\) be a set at which Sen's \(\alpha\) doesn’t hold. That is, there is an \(x \in C(B)\) and \(A \subseteq B\) with \(x \in A \setminus C(A)\). Let \(A^*\) be a maximal (w.r.t set inclusion) such counterexample (in \(B\)). Let \(y \in B \setminus A^*\). Note that (i) \(y\) exists as \(A^* \neq B\), and (ii) \(y \neq x\) as \(x \in A^*\). By maximality of \(A^*\), \(x \in C(A^* \cup \{y\})\). Thus, \(A^* \cup \{y\} \succ (A^* \cup \{y\}) \setminus x\). By the contrapositive of \(A2^*\) (Modularity), this implies \(A^* \setminus x \not\sim A^*\). Thus, by \(A1^*\), \(A^* \not\sim A^* \setminus x\) so that \(x \in C(A^*)\) – a contradiction.

**Proof of Proposition 3.**

**Step 1.** We prove \((\succeq_1 \text{ MPF} \succeq_2) \Rightarrow \{B^1_i\}\) coarsens \(\{B^2_i\}\) via contraposition. Take a common labeling of the partitions such that \(B^1_i \succ_1 B^1_j\) whenever \(i < j\) (for \(r = 1, 2\)). Assume that \(\{B^1_i\}\) is not a coarsening of \(\{B^2_i\}\). Then, there is an \(i^*\) and a collection \(\{j_1, \ldots, j_n\}\) \((n \geq 2)\) such that \(B^2_{j_i} \subseteq \bigcup_{k=1}^n B^1_{j_k}\) and \(B^2_{j_i} \not\subseteq \bigcup_{k \in S} B^1_{j_k}\), for any \(S \subseteq \{1, \ldots, n\}\). Put \(A := B^2_{j_1} \cap B^1_{j_1}\), \(B := B^2_{j_2}\). Then, \(A \subsetneq B\) and \(A \sim_1 B\), yet \(B \succ_2 A\), implying \(\neg (\succeq_1 \text{ MPF} \succeq_2)\).
Step 2. Assume that \( \{B_i^1\} \) is a coarsening of \( \{B_i^2\} \) and choose a common labeling of the partitions so that \( B_i^1 \succ A_i^1 \) (resp. \( B_i^2 \succ B_i^2 \)) whenever \( i < j \). Let \( A \subseteq B \) and assume \( B \succ A \). Put \( i_A^1 := \min \{i : A \cap B_i^2 \neq \emptyset\} \), \( i_B^2 := \min \{i : B \cap B_i^2 \neq \emptyset\} \) and similarly define \( i_A^1, i_B^1 \). Notice that \( i_A^2 \geq i_B^2 \) as \( A \subseteq B \). We consider two cases.

Case (i). \( i_A^2 \geq i_B^2 \).

Since \( \{B_i^1\} \) is a coarsening of \( \{B_i^2\} \) and \( (\succeq_1)|_X = (\succeq_2)|_X \), we must have \( i_A^1 = i_B^1 \). Thus, we obtain

\[
B \sim_1 B \cap B_{i_B^1}^1 \succeq_1 A \cap B_{i_B^1}^1 = A \cap B_{i_A^1}^1 \sim_1 A.
\]

Since

\[
B \cap B_{i_B^2}^2 \sim_2 B \succ_2 A \cap B_{i_B^2}^2
\]

we have \( A \cap B_{i_A^1}^1 \subseteq B \cap B_{i_B^2}^2 \). This implies that \( A \cap B_{i_B^2}^2 \subseteq B \cap B_{i_A^1}^1 \), so that \( B \succ_1 A \), since \( \succeq_1 \in \Sigma_{LSPF} \).

Case (ii). \( i_A^1 > i_B^2 \).

Note that \( i_A^1 \geq i_B^2 \). If \( i_A^1 > i_B^2 \), then we clearly obtain \( B \succ_1 A \). Thus, assume \( i_A^1 = i_B^2 \).

Find the maximal \( j \) such that \( \cup_{i_B^2} B_i^2 \subseteq B_{i_A^1}^1 \). If \( j \geq i_A^1 \), then since \( i_A^1 > i_B^2 \) we know that \( A^* := A \cap (\cup_{i_B^2} B_i^2) \subseteq B \cap (\cup_{i_B^2} B_i^2) := B^* \). Since \( \succeq_1 \in \Sigma_{LSPF} \), \( B^* \succ_1 A^* \).

Moreover, \( A^* \sim_1 A, B^* \sim_1 B \) so that \( B \succ_1 A \). If \( j < i_A^1 \), then \( A \cap B_{i_B^2}^2 = \emptyset \) so that \( A \subseteq \cup_{k>i_B^2} B_k \). Since

\[
B^* \sim_1 (B^* \cup_{k>i_B^2} B_k^1) \sim_1 \cup_{k>i_B^2} B_k^1 \succeq_1 A
\]

and \( B^* \sim_1 B \), we obtain \( B \succ_1 A \). \( \Box \)

Proof of Theorem 4.

Step 1. We show that \( (\succeq_1 \ MPF \succeq_2) \Rightarrow ((\succeq_1) \text{ coarsens } (\succeq_2)) \). To show this we check that, menu-by-menu, the subjective relation \( \succeq_1 \) is a coarsening of the relation \( \succeq_2 \). Since we are assuming \( (\succeq_1 \ MPF \succeq_2) \), we immediately obtain the containment

\[
C^2(A) \subseteq C^1(A), \forall A \in \mathcal{M}
\]

where \( C^i(\cdot), C^2(\cdot) \) (resp.) denote the critical element correspondences for the two menu preferences. Moreover, since \( \{\succeq_1\} \) and \( \{\succeq_2\} \) are the (unique) most refined local orders representing (resp.) \( \succeq_1 \) and \( \succeq_2 \), we obtain (see (3))

\[
x \sim_A^i y \iff xR_A^i y
\]

where the superscript denotes agent \( i \). It follows that if \( x \sim_A^2 y \), then \( x, y \in C^2(A') \) for some \( A' \subseteq A \). Since \( C^2(A') \subseteq C^1(A') \), this implies \( xR_A^1 y \), so that \( x \sim_A^1 y \). Now
consider $x, y$ with $x \succ_A y$. We verify that $x \succeq_A y$. Since $\succeq_A$ coarsens the singleton ranking and $(\succeq_1)|_X = (\succeq_2)|_X$, i.e. the two menu preferences agree on the singleton ranking, we must have $\{x\} \succ \{y\}$. Hence, $x \succeq_A \{x,y\} \succ y$. By inductive consistency, it now follows that $x \succeq_A y$.

Step 2. We show that $(\{\succeq_1\} \text{ coarsens } \{\succeq_2\}) \Rightarrow (\succeq_1 \text{ MPF } \succeq_2)$. Notice that if $\{\succeq_1\} \text{ coarsens } \{\succeq_2\}$, then we must have $C^2(A) \subseteq C^1(A)$. To see this, note that $C^i(A)$ is the set of $\succeq_A$-maxima. Hence, if $\succeq_A$ coarsens $\succeq_2$, and $x \succeq_2$-maximal, then $x \succeq_A$-maximal. This implies that $C^2(A) \subseteq C^1(A)$. Now take $A \subseteq B$ and assume that $B \succ_A A$. We will show that $B \succ_A A$. Consider two sub-cases, (i) $C^2(B) \cap A \neq \emptyset$, or (ii) $C^2(B) \cap A = \emptyset$. In the former case, $A^2$ implies $C^2(A) \subseteq C^2(B)$. Since $C^2(B) \subseteq C^1(B)$ and $C^2(A) \neq \emptyset$ by $A3$, we obtain $C^1(B) \cap A \neq \emptyset$. Hence, $A^2$ implies $C^1(A) \subseteq C^1(B)$. We verify that $C^1(A) \subseteq C^1(B)$, which then implies $B \succ_A A$ since $C^1(B) \sim_1 B, C^1(A) \sim_1 A$ (by Step 1 in the proof of Theorem 1). Proceed via contradiction, and assume $C^1(B) = C^1(A)$. Since $B \succ_A A$, select an $x \in C^2(B) \setminus A$.\textsuperscript{36} Since $C^2(B) \subseteq C^1(B)$, we have $x \in C^1(B)$. OTOH, since $C^1(A) \subseteq A$, if $C^1(A) = C^1(B)$, then $x \in A$, which contradicts our choice of $x$. This proves the result in the case where $C^2(B) \cap A = \emptyset$. The second case is where $C^2(B) \cap A = \emptyset$. Consider two sub-cases (iia) $C^2(B) \cap A = \emptyset$ and $C^1(B) \cap A = \emptyset$, or (iib) $C^2(B) \cap A = \emptyset$ and $C^1(B) \cap A \neq \emptyset$. In the latter case, we must have (by $A2$) $C^1(A) \subseteq C^1(B)$. Moreover, since $C^2(B) \subseteq C^1(B)$ and $C^2(B) \cap A = \emptyset$ we have $C^1(A) \subseteq C^1(B)$. Hence, $B \succ_A A$. In the former case, we have $C^1(B) \cap A = \emptyset$. Hence, find $x \in C^1(B)$ (by $A3$) and note that $B \succ_1 B \setminus x \succeq_1 A$ – the latter relation by monotonicity ($A1^*$), implying that $B \succ_1 A$.\hfill \square

6.4 Miscellaneous Proofs

Lemma 1. $\text{DFC implies } A1$ and $\text{Set-Betweenness implies } A1$. Moreover, $\text{Set-Betweenness implies that } A \setminus x \succeq A$ for any $x \in \inf(A)$.

Proof. That $A1$ implies DFC is straightforward. We check that Set-Betweenness implies $A \setminus x \succeq A$ for any $x \in \inf(A)$. This verifies $A1$. Rank the elements of $A$ top to bottom: $A = \{x_1, \ldots, x_k\}$, where $\{x_1\} \succeq \{x_2\} \succeq \cdots \succeq \{x_k\}$. By Set-Betweenness, for any $l$, $\{x_{l-1}\} \succeq \{x_{l-1}, x_l\} \succeq \{x_l\}$. This yields the following chain: $\{x_1\} \succeq \{x_1, x_2\} \succeq \{x_2\} \succeq \{x_2, x_3\} \succeq \{x_3\} \succeq \cdots \succeq \{x_{k-1}\} \succeq \{x_{k-1}, x_k\} \succeq \{x_k\}$. Thus, $\{x_1, x_2\} \succeq \{x_2, x_3\} \succeq \cdots \succeq \{x_k-1, x_k\}$. Successive applications of Set-Betweenness ($k-1$ times) then yield: $\{x_1, x_2\} \succeq \{x_1, x_2, x_3\} \succeq \cdots \succeq \{x_1, x_2, \ldots, x_{k-1}\} \succeq \{x_1, x_2, \ldots, x_{k-1}, x_k\}$, so that $A \setminus x_k \succeq A$.\hfill \square

We call a utility $\mathcal{U} : \mathcal{M} \rightarrow \mathbb{R}$ a No-Aggregation ($\text{NAG}$) Dekel et al. (2009) utility if there is a $k + 1$-tuple $(u, v_1, \ldots, v_k)$ of functions and a probability distribution

\textsuperscript{36}Note that such an $x$ exists. Else, $C^2(B) \subseteq A$ so that $A \succeq_2 C^2(B) \sim_2 B$ – a contradiction.
\( q = (q_1, \ldots, q_k) \) such that: \( U(A) = \sum_{i=1}^{k} q_i \cdot \max_{x \in A} (u(x) + v_i(x)) - \max_{x \in A} v_i(x) \).

The more general model axiomatized in Dekel et al. (2009) takes the form, \( U(A) = \sum_{i=1}^{k} q_i \cdot [u(x) - \sum_{j=1}^{k_i} (\max_{x \in A} (v_j(z)) - v_j(z))] \), where the state-dependent terms admit aggregation of self-control costs. We check that Example 4 does not admit a NAG-DLR representation.

**Claim 2.** NAG-DLR cannot represent Example 4.

**Proof.** Proceed via contradiction. Let \( \{q_i\}, u, v \) be a representing triple and observe that \( \{b\} \sim \{b, y\} \succ \{y\} \) implies \( \max_{x \in \{b,y\}} (u(x) + v_i(x)) = u(b) + v_i(b), \forall i \). That is, \( b \) is chosen in every state. To see this, first note that \( \{b\} \succ \{y\} \) yields \( u(b) > u(y) \).

Thus, if there is an \( i \) such that \( \max\{u(b) + v_i(b), u(y) + v_i(y)\} = u(y) + v_i(y) \), then, for this \( i \), we must have \( v_i(y) > v_i(b) \) so that \( u(b) > \max_{x \in \{b,y\}} (u(x) + v_i(x)) - \max_{x \in \{b,y\}} v_i(x) = u(y) \).

Moreover, if \( \max_{x \in \{b,y\}} (u(x) + v_i(x)) = u(b) + v_i(b) \) we also have \( u(b) \geq u(b) + v_i(b) - \max_{x \in \{b,y\}} v_i(x) \). Since \( \{b\} \sim \{b, y\} \succ \{y\} \), we obtain: \( \max_{x \in \{b,y\}} (u(x) + v_i(x)) = u(b) + v_i(b), \forall i \).

On the other hand, we also know that

\[
C(A) \subseteq \bigcup_i \arg \max_{x \in A} (u(x) + v_i(x)).
\]

Since \( y \in C(\{b, y, c\}) \), there must be some \( i \) such that \( \{y\} = \arg \max_{x \in \{b,y,c\}} (u(x) + v_i(x)) \). But then, for this same \( i \), \( \{y\} = \arg \max_{x \in \{b,y\}} (u(x) + v_i(x)) \) – contradiction.

Recall the definition of the base relation associated to a choice correspondence: \( xR_C y \iff x \in \hat{C}([x, y]) \). Consider the following axioms on this correspondence.

**Sen’s \( \beta^+ \):** If \( A \subseteq B \) and \( \hat{C}(B) \cap A \neq \emptyset \), then \( \hat{C}(A) \subseteq \hat{C}(B) \).

**Base Transitivity:** \( R_C \) is a transitive relation.

The title “Sen’s \( \beta^+ \)” is taken from an exercise in Austen-Smith and Banks (1999), where this axiom appears. It is also used in Tyson (2008), under the title “Strong Expansion”. Note that if \( \hat{C}() \) admits a rationalization by an inductively consistent system, then \( \hat{C}() \) satisfies Sen’s \( \beta^+ \). If we assume transitivity of the base relation, then we also get a converse. A related result is obtained in Tyson (2008). However, we are not providing a characterization – Tyson (2008) provides a characterization result. In particular, Tyson shows that Sen’s \( \beta^+ \) characterizes choice functions which are rationalized by a family of relations \( \{R_A\} \) satisfying a nestedness criterion. Nestedness is the right-to-left (\( \Rightarrow \)) direction of the equivalence: \( x \succeq_A y \iff x \succeq_B y \) (when \( A \subseteq B \)) which defines deductive consistency. Hence, inductive consistency and nestedness comprise the two directions that define this equivalence.

**Lemma 2.** If \( \hat{C}() \) satisfies Sen’s \( \beta^+ \) and Base Transitivity, then \( \hat{C}() \) admits a context-dependent rationalization. Conversely, if \( \hat{C}() \) admits a context-dependent rationalization, then \( \hat{C}() \) satisfies Sen’s \( \beta^+ \).
Proof. For necessity of Sen’s $\beta+$ assume that $x \in \hat{C}(A)$ and that $z \in \hat{C}(B) \cap A \neq \emptyset$. Then, by representability, $x \in \Phi_u(A)$, so that $x \succeq_A z$. By inductive consistency, this implies that $x \succeq_B z$. Since $z \in \Phi_u(B)$, this implies $x \in \Phi_u(B)$. Hence, $\hat{C}(A) \subseteq \hat{C}(B)$. For sufficiency, we first define an auxiliary system of equivalence relations $\{\mathcal{R}_A\}$, which we call the “not separated” relation. Put

$$x \mathcal{R}_A y \iff \exists A' \subseteq A, \text{ s.t. } x, y \in \hat{C}(A')$$

We first claim that $\mathcal{R}_A$ is an equivalence relation on $A$. The relation is clearly (i) well-defined and (ii) symmetric. We check transitivity. Take $(x, y, z)$ with $x \mathcal{R}_A y, y \mathcal{R}_A z$. Let $A_1, A_2$ be menus such that $\{x, y\} \subseteq \hat{C}(A_1), \{y, z\} \subseteq \hat{C}(A_2)$. Consider the menu $A' = A_1 \cup A_2$ and note that $\hat{C}(A') \neq \emptyset$. Hence, either $\hat{C}(A') \cap A_1 \neq \emptyset$ or $\hat{C}(A') \cap A_2 \neq \emptyset$. If the former holds, then $\{x, y\} \subseteq \hat{C}(A_1) \subseteq \hat{C}(A')$ by Sen’s $\beta+$. Hence, $y \in \hat{C}(A') \cap A_2 \neq \emptyset$ and we obtain $\{y, z\} \subseteq \hat{C}(A_2) \subseteq \hat{C}(A')$. Hence, $(x, z) \subseteq \hat{C}(A')$, implying that $x \mathcal{R}_A z$. Similarly, argue if $\hat{C}(A') \cap A_2 \neq \emptyset$, then $\hat{C}(A_2) \subseteq \hat{C}(A')$ which implies $(x, z) \subseteq \hat{C}(A')$, so that $x \mathcal{R}_A z$. Hence, the binary relation $\mathcal{R}_A$ is transitive. It follows that the $\mathcal{R}_A$-equivalence classes form a partition of $A$. We use the base relation to put an ordering on these equivalence classes. This will yield the local orders $\{\succeq_A\}$.

Since we are assuming Base Transitivity and the base relation is complete, it is an order on $X$. Let $\succeq$ denote this order. Notice that if $x, y \in A$ where $x \succeq_A y$, then $y \in \hat{C}(A)$ implies $x \in \hat{C}(A)$, by Sen’s $\beta+$. Hence, the $\mathcal{R}_A$-indifference classes are $\succeq$-order intervals in $A$. This defines an ordering $\succeq_A$ (that coarsens $\succeq$). It remains to check that (i) $\hat{C}(A) = \Phi_u(A)$ and (ii) $\{\succeq_A\}$ is inductively consistent. For the latter, let $A \subseteq B$ with $x \succeq_A y$. Note that, by definition, $x \sim_A y$ if $x \mathcal{R}_A y$. Since $x \mathcal{R}_A y$ implies that $x \mathcal{R}_B y$ (when $A \subseteq B$), it follows that $x \sim_B y$. If $x \succ_A y$, then $x \succ y$. Hence, $x \succeq_B y$, as $\succeq_B$ coarsens the base relation $\succeq$. This verifies inductive consistency. For (i), note that the $\succeq_A$-classes are order intervals. Moreover, if $x \succ_A y$, then $y \notin \hat{C}(A')$ for any $A' \subseteq A$ with $x, y \in A'$. It follows that $\Phi_u(A) \subseteq \hat{C}(A)$. For the reverse inclusion, note that $x \mathcal{R}_A y, \forall x, y \in \hat{C}(A)$. Hence, $z \in \Phi_u(A)$ and note that $x \sim_A z, \forall x \in \hat{C}(A)$. 

\[\square\]
References


